Lecture 17 Inverse fcn thm, Local mapping thm

HWK 5 due Thurs, March 3

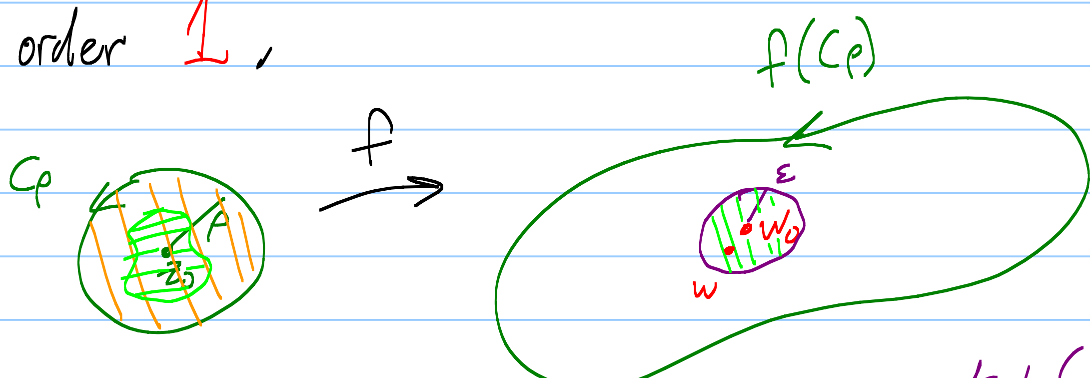
Super inverse fcn thm. Suppose f analytic on a domain Ω_1 and one-to-one on Ω_1. Then $\Omega_2 = f(\Omega_1)$ is a domain and $f^{-1}: \Omega_2 \to \Omega_1$ is analytic. Furthermore, f' is nonvanishing on Ω_1 (and $(f^{-1})'$ is nonvanishing too). $(f^{-1})'(w) = \dfrac{1}{f'(f^{-1}(w))}$

"Locally 1-1 analytic fcns have nonvanishing derivatives."

Inverse fcn thm: If f is analytic on $D_\delta(z_0)$ and $f'(z_0) \neq 0$, $\exists\ \varepsilon > 0$ and a domain $U = f^{-1}(D_\varepsilon(w_0))$ $[w_0 = f(z_0)]$ such that $f: U \xrightarrow[\text{onto}]{1\text{-}1} D_\varepsilon(w_0)$ and f^{-1} is analytic on $D_\varepsilon(w_0)$. Chain rule: $f'(f^{-1}(w))(f^{-1})'(w) = 1$

Pf: $f(z_0) = w_0$, $f'(z_0) \neq 0$. So $f(z) - w_0$ is not constant. So zero of $f(z) - w_0$ at z_0 is isolated.

Pick ρ with $0 < \rho < \delta$ such that z_0 is only zero of $f(z) - w_0$ in $\overline{D_\rho(z_0)}$. $f'(z_0) \neq 0 \implies z_0$ is a zero of order 1.



$\varepsilon < \operatorname{dist}(w_0, f(C_\rho))$

$$\begin{pmatrix}\text{\# zeroes of} \\ f(z)-w \\ \text{inside } C_\rho\end{pmatrix} = \frac{1}{2\pi i}\int_{C_\rho} \frac{f'(z)}{f(z)-w}\,dz = H(w)$$

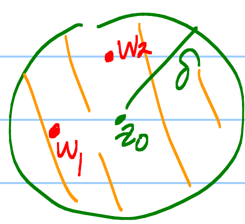
H is analytic on $D_\varepsilon(w_0)$ and <u>integer</u> valued there.

So $H(w) \equiv H(w_0) = 1$. Every $w \in D_\varepsilon(w_0)$ gets hit <u>exactly</u> once. Let $U = f^{-1}(D_\varepsilon(w_0))$.

U is open because f cont. OMT $\Rightarrow$ f^{-1} cont. $\Rightarrow$ U is a domain. Use Super IFT to see f^{-1} analytic.

<u>Another way to see f is locally 1-1 if $f'(z_0) \neq 0$.</u>

$f'(z) = f'(z_0) + E(z)$ where $E(z) \to 0$ as $z \to z_0$,
so $\exists\ \delta > 0$ such that $|E(z)| < \varepsilon < \frac{|f'(z_0)|}{2}$ on $\overline{D_\delta(z_0)}$.



$$f(w_2) - f(w_1) = \int_{L_{w_1}^{w_2}} f'(z)\,dz$$

$$= \int_{L_{w_1}^{w_2}} f'(z_0) + E(z)\,dz$$

$$= f'(z_0)(w_2 - w_1) + \underbrace{\int_{L_{w_1}^{w_2}} E(z)\,dz}_{\mathcal{E}}$$

$$|f(w_2) - f(w_1)| \geq \Big|\, \underbrace{|f'(z_0)(w_2 - w_1)|} - |\mathcal{E}| \,\Big|$$

$|f'(z_0)|\,|w_2-w_1|$ $\qquad |\mathcal{E}| \le \left(\underset{L_{w_1}^{w_2}}{\text{Max}}|\mathcal{E}|\right)|w_2-w_1|$

$$< \varepsilon \cdot |w_2-w_1|$$

$$\ge |f'(z_0)|\,|w_2-w_1| - \frac{|f'(z_0)|}{2}|w_2-w_1| \qquad < \frac{|f'(z_0)|}{2}|w_2-w_1|$$

$$\ge \frac{|f'(z_0)|}{2}|w_2-w_1| \longleftarrow \text{not zero if } w_2 \ne w_1.$$

So f is 1-1. Use Super IFT to see f^{-1} is analytic.

Local mapping theorem. $f(z_0)=w_0$, f analytic, not const.

Then zero of $f(z)-w_0$ at z_0 is isolated. Get $D_\rho(z_0)$

is only zero on $\overline{D_\rho(z_0)}$.

Def^n: Say $f(z_0)=w_0$ with mult N if order of zero at $f(z)-w_0$ is N.

$$f(z)-w_0 = (z-z_0)^N F(z), \quad F \text{ analytic}, \ F(z_0)\ne 0.$$

F nonvanishing on convex $D_\rho(z_0)$. So $\exists$ analytic

N-th root of F: $g(z)^N = F(z)$ on $D_\rho(z_0)$.

$$f(z)-w_0 = \Big[\underbrace{(z-z_0)\,g(z)}_{G(z)}\Big]^N$$

Notice that $G(z_0)=0$.

Claim: $G'(z_0)\ne 0$

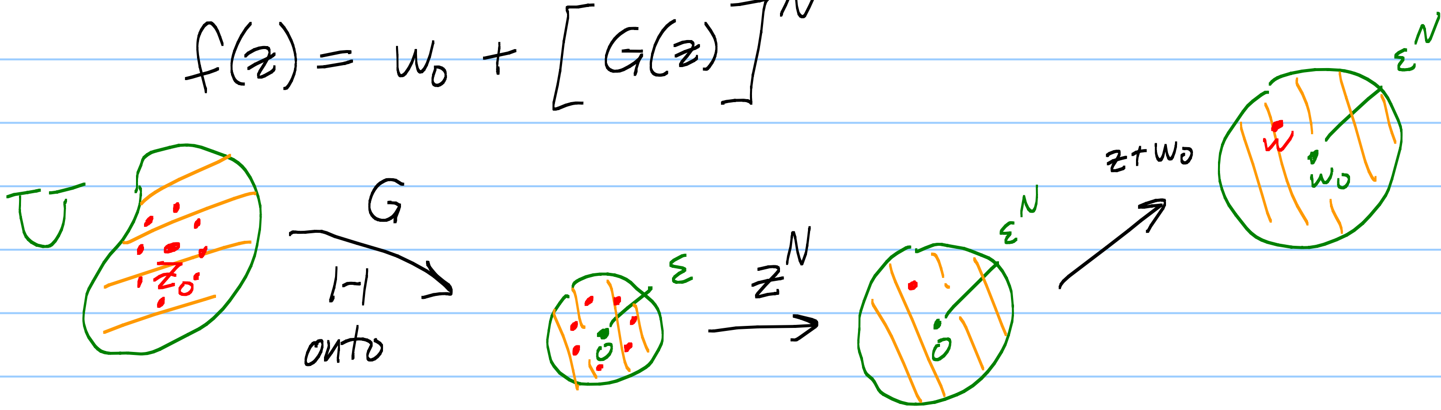
$$G'(z) = 1\cdot g(z) + (z-z_0)\,g'(z)$$

$$G'(z_0) = g(z_0) + 0 \neq 0 \quad (\text{because } F(z_0) \neq 0)$$

Get $D_\varepsilon(0)$ and U from IFT such

$$G : U \xrightarrow[\text{onto}]{1\text{-}1} D_\varepsilon(0), \quad G' \text{ nonvanishing}, \ G^{-1} \text{ analytic.}$$

$$f(z) = w_0 + [G(z)]^N$$



$f : U - \{z_0\} \longrightarrow D_{\varepsilon^N}(w_0) - \{w_0\}$ is N-to-one

$f^{-1}(w) = \{N$ <u>distinct</u> pts in U of mult one$\}$

if $w \neq w_0$. $\qquad f^{-1}(w_0) = \{z_0\}$.

f is an "N-sheeted covering map" of $U - \{z_0\}$ onto $D_{\varepsilon^N}(w_0)$. $\quad f : U \to D_{\varepsilon^N}(w_0)$ is a "branched covering map"

<u>Notice</u>: f is locally 1-1 $\iff N = 1$.

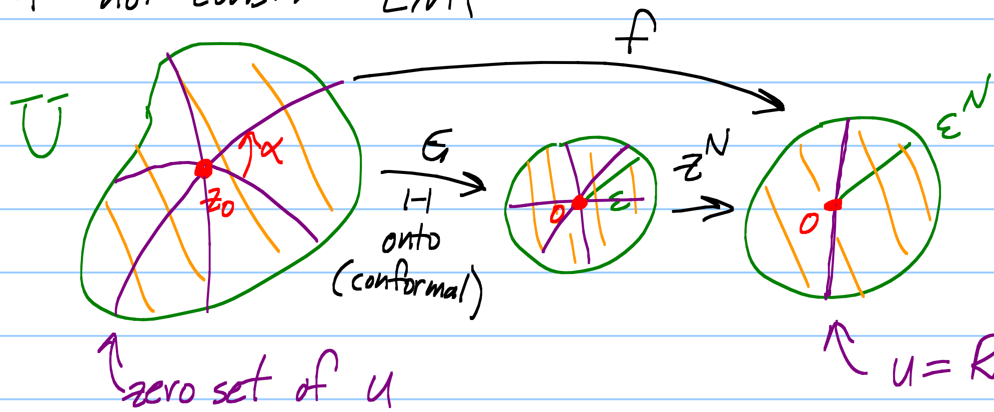
(Local mapping thm $\Rightarrow$ Super IFT and OMT.)

<u>Cor of LMT</u>: Zeroes of nonconstant harmonic <u>cannot</u> be isolated. Given locally as curves that cross at angles $2\pi/N$.

Pf: $w_0 = 0$ in LMT. $f = u + iv$

harm harm conj

f not const. LMT



Conformality $\Rightarrow$ angles $= \frac{2\pi}{N}$.