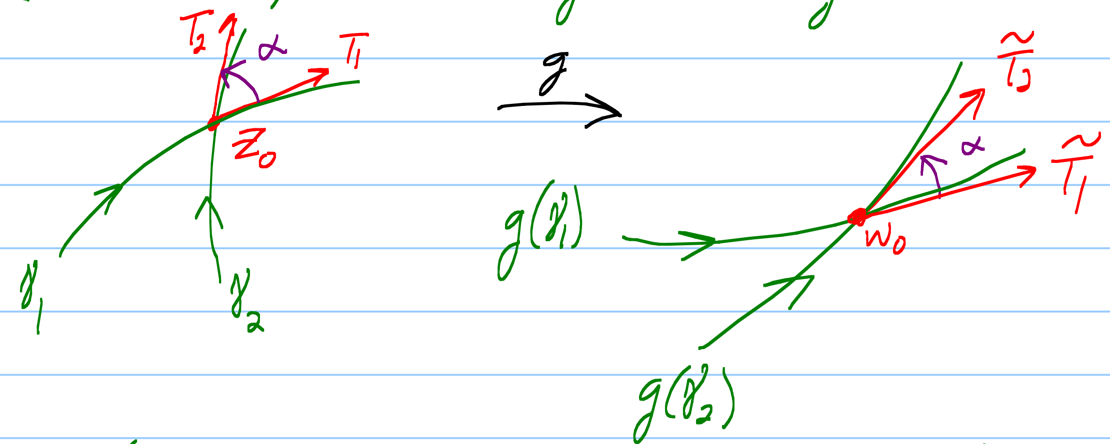
Lecture 18 Conformal mapping, Baby Residue theorem on Toy domains

HWK 5 due Thurs March 3 in GS

Big fact: Analytic g that is locally one-to-one preserves angles (is conformal). Note: g 1-1 $\Rightarrow$ g' nonvanishing.



$\gamma_j : z_j(t) \qquad z_j(0) = z_0 \qquad j = 1, 2. \qquad T_j = z_j'(0)$

$g(\gamma_j) : g(z_j(t)) \qquad w_0 = g(z_0) \qquad \tilde{T}_j = g'(z_j(t))\, z_j'(t) \Big|_{t=0}$

$$\boxed{\tilde{T}_j = \underbrace{g'(z_0)}_{re^{i\theta}} T_j}$$

stretch by factor of $r > 0$, rotate counterclockwise by θ. ✓

Converse is "almost" true:

$f(x+iy) = u(x,y) + i v(x,y)$. Suppose u, v C^1-smooth and $\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \neq 0$.

If f is conformal, then f is analytic.

Pf: $J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \underset{\text{analytic}}{=} \begin{bmatrix} R\cos\theta & -R\sin\theta \\ R\sin\theta & R\cos\theta \end{bmatrix}$

causes rotation by θ — geom — see C-R equs!

MA 525 $\quad f(x+iy) = u + iv$

Def^n: Analytic : u, v $\mathcal{C}^1$-smooth satisfy C-R equs.

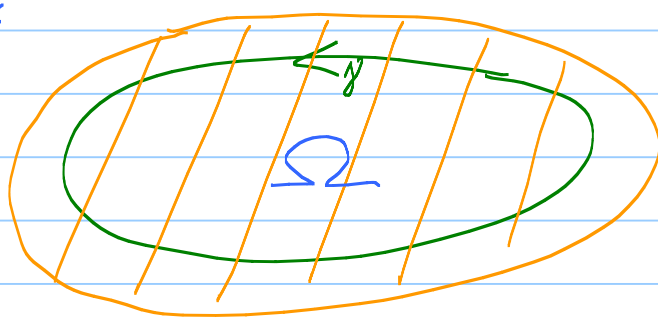
I.F.T. $\quad \mathbb{J} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$

$$\det \mathbb{J} = u_x^2 + v_x^2 = |f'|^2$$

$$f'(z_0) \neq 0 \implies \det \mathbb{J} \neq 0, \quad \mathbb{R}^2 \to \mathbb{R}^2 \text{ I.F.T.}$$

$$\mathbb{J}_{f^{-1}} = (\mathbb{J}_f)^{-1} = \underbrace{\frac{1}{A^2+B^2} \begin{bmatrix} A & B \\ -B & A \end{bmatrix}}_{\text{See C-R equs!}}$$

Cauchy Theorem:



f analytic

Ω = "inside of γ"

$$\int_\gamma f\,dz = \int_a^b \left[u(x(t), y(t)) + i v(x(t), y(t))\right] \cdot \left[\dot{x}(t) + i \dot{y}(t)\right] dt$$

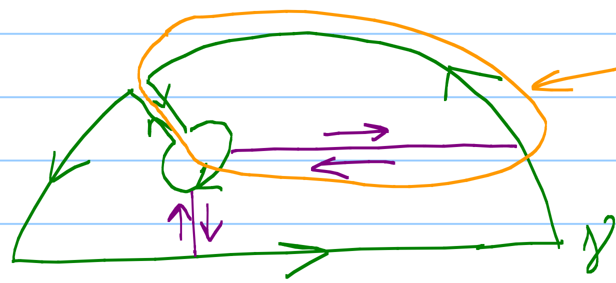
$$= \int_a^b (u\dot{x} - v\dot{y})\,dt + i \int_a^b (u\dot{y} + v\dot{x})\,dt$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx$$

$$\underset{\substack{\uparrow \\ \text{Green's}}}{=} \iint_\Omega \underbrace{-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}}_{\substack{=0 \\ \text{CR-eqn \#2}}} dA + i \iint_\Omega \underbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}_{\substack{=0 \\ \text{CR-eqn \#1}}} dA$$

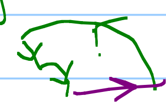
$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

<u>Back to MA 530</u> Stein : Toy regions



Cauchy on convex

$$\int f\,dz = 0$$



Add up. See Cauchy thm.

Thm by definition: Cauchy thm holds on Toy regions.

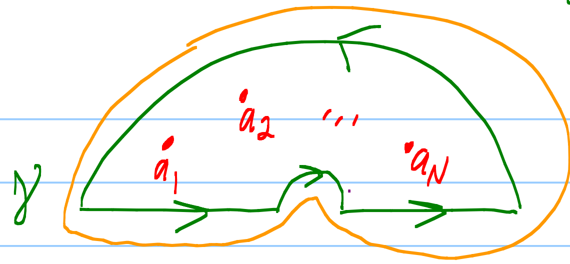
<u>Def^n</u>: If f has a pole at a, we know that

near a,
$$f(z) = \underbrace{\frac{A_N}{(z-a)^N} + \cdots + \frac{A_1}{z-a}}_{\text{Princ part}} + (\text{analytic})$$

$A_1 \leftarrow \text{Res}_a f$

A_1 = Residue of f at a.

Baby Residue thm on Toy regions



f is analytic "inside and on" γ except at finitely many poles inside γ.

$$\int_\gamma f \, dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{a_j} f = 2\pi i \left(\begin{array}{c} \text{Sum of residues} \\ \text{of } f \text{ inside } \gamma \end{array} \right)$$

Pf: $f - \sum_{j=1}^{N} R_j$ has removable singularities at each a_j.

Princ part of f at a_j

Give it values at a_j's to make it analytic.

Cauchy thm $\quad 0 = \int_\gamma f - \sum_1^N R_j \; dz$

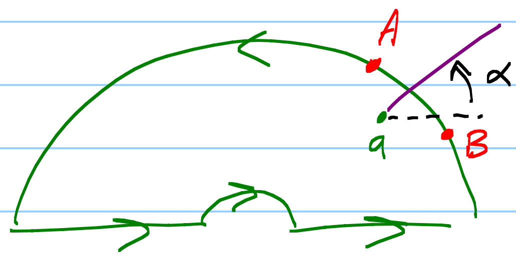
$$\int_\gamma f \, dz = \sum_1^N \int_\gamma R_j \, dz$$

Aha! $\quad \int_\gamma \frac{1}{(z-a)^n} \, dz = 0 \quad \text{if} \quad n \neq 1$

$$\frac{d}{dz}\left[\frac{1}{-n+1} (z-a)^{-n+1} \right] = (z-a)^{-n}.$$

Key: $\displaystyle\int_\gamma \frac{1}{z-a}\,dz = \begin{cases} 2\pi i & a \text{ inside } \gamma \\ 0 & a \text{ outside} \end{cases}$

$= H(a)$ Cauchy thm



Pick a close to γ so ray cuts γ at an angle in exactly one place.

$$\int_\gamma \frac{1}{z-a}\,dz = \lim_{\substack{A \text{ slides down} \\ B \text{ slides up}}} \int_{\gamma_A^B} \underbrace{\frac{1}{z-a}}_{\frac{d}{dz}\log_\alpha(z-a)}\,dz$$

$$= \lim \log_\alpha(B-a) - \log_\alpha(A-a)$$

$$= \lim \Big[\big(\ln|B-a| + i(\alpha+2\pi-\varepsilon)\big) - \big(\ln|A-a| + i(\alpha+\varepsilon)\big)\Big]$$

$$= 2\pi i \quad \text{as } A\downarrow,\ B\uparrow$$

$H(w) = \displaystyle\int_\gamma \frac{1}{z-w}\,dz$ is analytic in w.

$$H'(w) = \int_\gamma \frac{1}{(z-w)^2}\,dz = \int_\gamma \frac{d}{dz}\left[\frac{-1}{z-w}\right]dz \equiv 0$$

$H(a) = 2\pi i$. So $H \equiv 2\pi i$ inside γ.

Only $\dfrac{A_1}{z-a_j}$ terms contribute: $A_1\, 2\pi i$ ✓

<u>Fact</u>: $\frac{d}{dz}\log_\alpha z = \frac{1}{z}$

because $e^{\log_\alpha z} = z$

Chain rule: $\underbrace{e^{\log_\alpha z}}_{z} \frac{d}{dz}(\log_\alpha z) = 1$ ✓