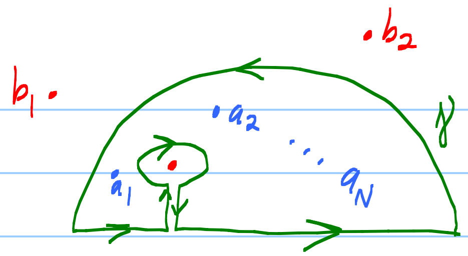
Lecture 19 Fun with the Baby Residue Theorem

HWK 5 due next Thurs, March 3 in GS

Midterm Exam: Fri after Spring break



$$\int_\gamma f\,dz = 2\pi i \sum_{j=1}^{N} \text{Res}_{a_j} f$$

f has finitely many poles in a Toy region enclosed by γ

Thm: Suppose P, Q polynomials and

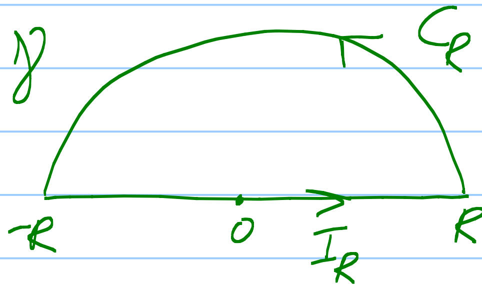
1) $\deg Q \geq \deg P + 2$

2) Q has no zeroes on $\mathbb{R}$

Then $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a \in Z_Q^+} \text{Res}_a \frac{P}{Q}$$

where $Z_Q^+ = \{\text{zeroes of } Q \text{ in the UHP}\}$ ← upper half plane

Pf:



$I_R : z(t) = t, \quad -R \leq t \leq R$

$z'(t) = 1$

$$\int_{I_R} \frac{P(z)}{Q(z)}\,dz = \int_{-R}^{R} \frac{P(t)}{Q(t)} \cdot 1\,dt \leftarrow \text{want}$$

Take R big enough that γ contains Z_Q^+.

Claim: $\displaystyle\int_{C_R} \frac{P}{Q}\,dz \to 0$ as $R \to \infty$.

Key: Basic poly estimate

$$a|z|^{\deg P} \le \underline{|P(z)| \le A|z|^{\deg P}}, \quad |z| > R_P$$

$$\underline{b|z|^{\deg Q} \le |Q(z)|} \le B|z|^{\deg Q}, \quad |z| > R_Q$$

So $$\left|\frac{P(z)}{Q(z)}\right| \le \frac{A}{b} \frac{1}{|z|^{\underbrace{\deg Q - \deg P}_{\ge 2}}} \le \frac{A}{b}\frac{1}{|z|^2} \qquad |z| > \mathrm{Max}(R_P, R_Q, 1)$$

$$\left(\text{Note } \frac{1}{|z|^{2+n}} < \frac{1}{|z|^2} \quad \text{when} \quad |z| > 1.\right)$$

Now $$\left|\int_{C_R} \frac{P}{Q}\, dz\right| \le \underbrace{\left(\mathrm{Max}_{C_R} \left|\frac{P}{Q}\right|\right)}_{\frac{A}{b}\frac{1}{R^2}} \cdot \underbrace{\mathrm{Length}(C_R)}_{\pi R}$$

$$\longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

Res Thm $$\int_{I_R} + \int_{C_R} = 2\pi i \sum \mathrm{Res}$$

$$\downarrow \qquad \downarrow$$

$$\int_{-\infty}^{\infty} + 0 = \checkmark$$

EX: $$I = \int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$$

$z^4 + 1 = 0$

Roots $\underbrace{e^{i\pi/4}, e^{i3\pi/4}}_{\text{inside}}, \underbrace{e^{i5\pi/4}, e^{i7\pi/4}}_{\text{outside}}$

$$z^4 + 1 = (z - r_1)(z - r_2)(z - r_3)(z - r_4)$$

Near r_1: $\dfrac{1}{z^4+1} = \dfrac{1}{z-r_1}\left[\underbrace{\dfrac{1}{(z-r_2)(z-r_3)(z-r_4)}}_{A_0+A_1(z-r_1)+A_2(z-r_1)^2+\cdots}\right]$

$$= \frac{A_0}{z-r_1} + \left(\begin{array}{c}\text{conv}\\ \text{power}\\ \text{series}\end{array} \text{ near } r_1\right)$$

where $\text{Res}_{r_1} = A_0 = \dfrac{1}{(r_1-r_2)(r_1-r_3)(r_1-r_4)} \leftarrow$ Ugh!

<u>Fact</u>: Suppose f, g analytic near a and g has a <u>simple</u> zero at a [meaning $g(a)=0$, $g'(a)\neq 0$].

Then $\boxed{\text{Res}_a \dfrac{f}{g} = \dfrac{f(a)}{g'(a)}}$

<u>Why</u>: $g(z) = \underset{\substack{\uparrow\\ g(a)=0}}{a_0} + \underset{\substack{\uparrow\\ \frac{g'(a)}{1!}\neq 0}}{a_1}(z-a) + a_2(z-a)^2 + \cdots$

$$= (z-a)\Big[\underbrace{a_1 + a_2(z-a) + \cdots}_{G(z)}\Big] \qquad G(a) = a_1 = \frac{g'(a)}{1!}$$

$$\frac{f(z)}{g(z)} = \frac{1}{z-a}\left[\underbrace{\frac{f(z)}{G(z)}}_{A_0+A_1(z-a)+\cdots}\right]$$

$$= \frac{A_0}{z-a} + \left(\begin{array}{c}\text{conv pow}\\ \text{series}\end{array}\right)$$

So $\quad \operatorname{Res}_a \frac{f}{g} = A_0 = \frac{f(a)}{G(a)} = \frac{f(a)}{g'(a)}$ ✓

Think about case: g has a double zero at a.

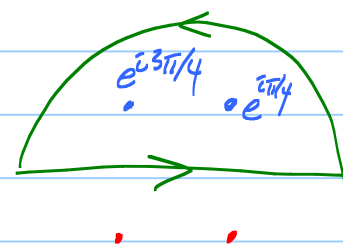
Now, $\quad I = 2\pi i \left(\operatorname{Res}_{e^{i\pi/4}} + \operatorname{Res}_{e^{i3\pi/4}} \right) \qquad \frac{1}{z^4+1} \begin{array}{l} \leftarrow f(z) \\ \leftarrow g(z) \end{array}$

$g(e^{i\pi/4}) = 0$ ✓

$g'(e^{i\pi/4}) = 4(e^{i\pi/4})^3 \neq 0$ ✓

$$= 2\pi i \left[\frac{1}{4e^{i3\pi/4}} + \frac{1}{4e^{i9\pi/4}} \right]$$

$$= \frac{1}{2}\pi i \left[\underbrace{e^{-i3\pi/4}}_{-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} + \underbrace{e^{-i9\pi/4}}_{\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} \right] = \frac{\pi}{\sqrt{2}}$$



<u>Fourier transform:</u> $\quad \hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

<u>EX:</u> $\quad I(s) = \int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\, dx$

<u>Case $s>0$</u>:

$$= 2\pi i \operatorname{Res}_i \frac{e^{isz}}{z^2+1}$$

$$= 2\pi i \frac{e^{is(i)}}{2(i)} = \pi e^{-s}$$

because

$$e^{isz} = e^{is(x+iy)} = e^{isx}\,e^{-sy}$$

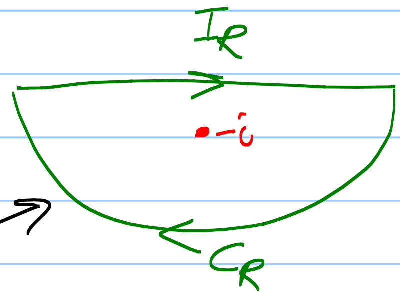
So $\left|e^{isz}\right| = e^{-sy} \le 1$ when $\begin{matrix} s>0 \\ y\ge 0 \end{matrix} \leftarrow$ UHP

and $|z^2+1| \ge \big||z^2|-1\big| \ge R^2-1$ if $|z|=R>1$.

So $\left|\int_{C_R} \frac{e^{isz}}{z^2+1}\,dz\right| \le \frac{1}{R^2-1}\cdot \pi R \to 0$ as $R\to\infty$.

Case $s<0$:

clockwise!

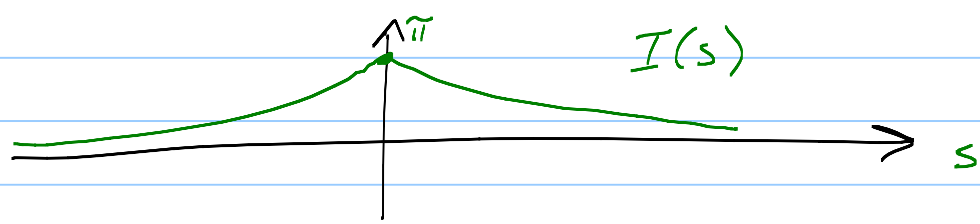


$\left|e^{isz}\right| = e^{-sy} \le 1$

$\begin{matrix} s<0 \\ y\le 0 \end{matrix} \leftarrow$ LHP

$$I(s) = \underset{\text{clockwise}}{-}\,2\pi i \operatorname{Res}_{-i} \frac{e^{isz}}{z^2+1}$$

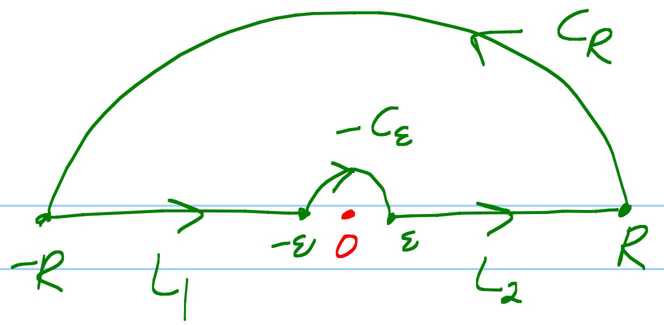
$$= -2\pi i \,\frac{e^{is(-i)}}{2(-i)} = \pi e^{s}$$



Next: $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$

$f(z) \overset{?}{=} \frac{\sin z}{z}$ No!

Trick: $\displaystyle \frac{e^{ix}}{x} = \frac{\cos x}{x} + i\,\frac{\sin x}{x}$



$$f(z) = \frac{e^{iz}}{z}$$

$$\int_L = \left(\int_{L_1} + \int_{L_2} \right) \frac{e^{iz}}{z}\, dz$$

$$= \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \underbrace{\frac{\cos t}{t}}_{\text{odd}} + i \underbrace{\frac{\sin t}{t}}_{\text{even}} \; dt$$

$\int$'s cancel

$$= 2i \int_{\varepsilon}^{R} \frac{\sin t}{t}\, dt \longleftarrow \text{want}$$

Cauchy's thm: $\displaystyle\int_\gamma \frac{e^{iz}}{z}\, dz = 0$

Claim: $\displaystyle\int_{C_R} \frac{e^{iz}}{z}\, dz \to 0$ as $R \to \infty$. (Jordan's lemma)

$$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\, dz \longrightarrow ? \quad \text{as} \quad \varepsilon \to 0.$$

$$\frac{e^{iz}}{z} = \frac{1 + (iz) + \frac{(iz)^2}{2!} + \cdots}{z} = \frac{1}{z} + \underbrace{\left(i - \frac{1}{2!} z + \cdots \right)}_{\text{entire!}}$$

$$= \frac{1}{z} + H(z) \longleftarrow |H| < M \text{ on } \overline{D_1(0)}$$