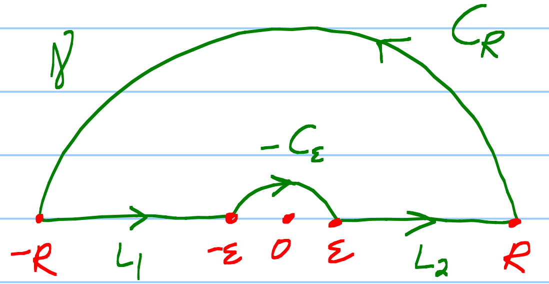
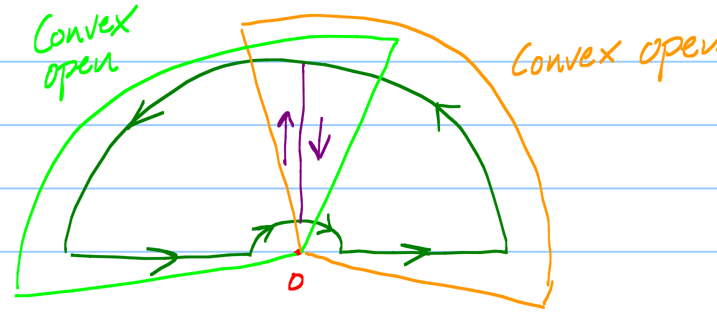
Lecture 20

$$\int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = \pi$$

HWK 5 due Thurs





$$0 = \int_\gamma \frac{e^{iz}}{z}\, dz = \underbrace{\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \underbrace{\frac{\cos t}{t}}_{\text{odd}} + i \underbrace{\frac{\sin t}{t}}_{\text{even}}\, dt}_{2i\int_\varepsilon^R \frac{\sin t}{t}\, dt} + \underbrace{\int_{-C_\varepsilon} \frac{e^{iz}}{z}\, dz}_{-\int_{C_\varepsilon} \frac{e^{iz}}{z}\, dz \;\to\; -i\pi \text{ as } \varepsilon\to 0} + \underbrace{\int_{C_R} \frac{e^{iz}}{z}\, dz}_{\to 0 \text{ as } R\to\infty \text{ via Jordans lemma}}$$

First, let $\varepsilon \to 0$. Then let $R \to \infty$.

Near 0, $$\frac{e^{iz}}{z} = \frac{1 + (iz) + \frac{(iz)^2}{2!} + \cdots}{z}$$

$$= \frac{1}{z} + \underbrace{\left(i - \frac{1}{2!} z - \frac{i}{3!} z^2 + \cdots\right)}_{H(z) \text{ entire!}}$$

$|H|$ bndd by M on $\overline{D_1(0)}$

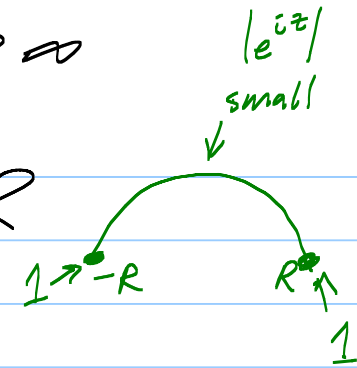
$$\int_{C_\varepsilon} \frac{e^{iz}}{z}\, dz = \underbrace{\int_{C_\varepsilon} \frac{1}{z}\, dz}_{\int_0^\pi \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\, dt} + \underbrace{\int_{C_\varepsilon} H\, dz}_{\to 0 \text{ as } \varepsilon \to 0}$$

$$= \int_0^\pi i\, dt = i\pi$$

$$\left|\int_{C_\varepsilon} H\, dz\right| \le \underbrace{\left(\max_{C_\varepsilon} |H|\right)}_{<M,\ \varepsilon<1} \cdot \pi\varepsilon$$

$$\le M\pi\varepsilon \to 0$$

Jordan's lemma: $\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| \to 0$ as $R \to \infty$



Basic est fails us: $\left| \quad \right| \le \left(\max_{C_R} \frac{|e^{iz}|}{R} \right) \pi R$

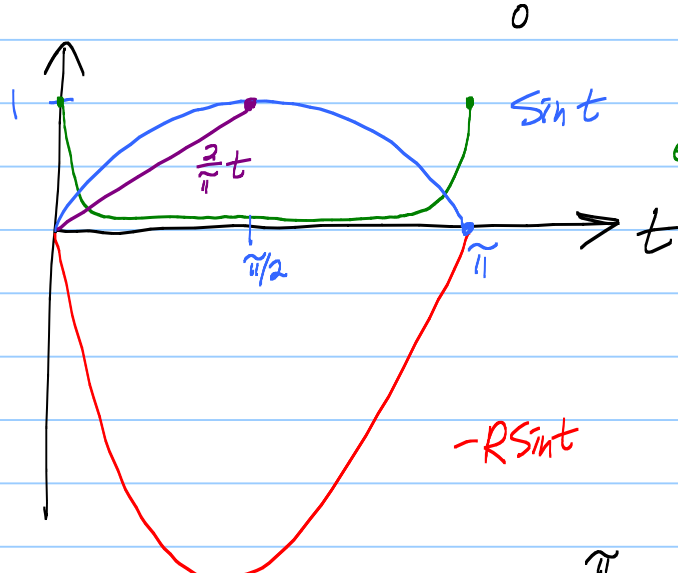
$$\left| e^{i(x+iy)} \right| = \left| e^{ix-y} \right| = e^{-y}$$

$$\le \frac{1}{R} \cdot \pi R = \pi \quad \text{Ouch!}$$

$$\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| = \left| \int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}}\, iRe^{it}\, dt \right|$$

$$= \left| i \int_0^{\pi} \underbrace{e^{iR\cos t - R\sin t}}_{e^{iR\cos t}\, e^{-R\sin t}}\, dt \right|$$

$$\le \int_0^{\pi} \left| \text{eee} \right| dt = \int_0^{\pi} e^{-R\sin t}\, dt$$



$$0 \le \frac{2}{\pi}t \le \sin t \le 1 \quad [0, \tfrac{\pi}{2}]$$

$$-R\sin t \le -\frac{2R}{\pi}t \le 0$$

$$0 < e^{-R\sin t} \le e^{-\frac{2R}{\pi}t} \le 1$$

So $$\int_0^{\pi} e^{-R\sin t}\, dt = 2 \int_0^{\pi/2} e^{-R\sin t}\, dt$$

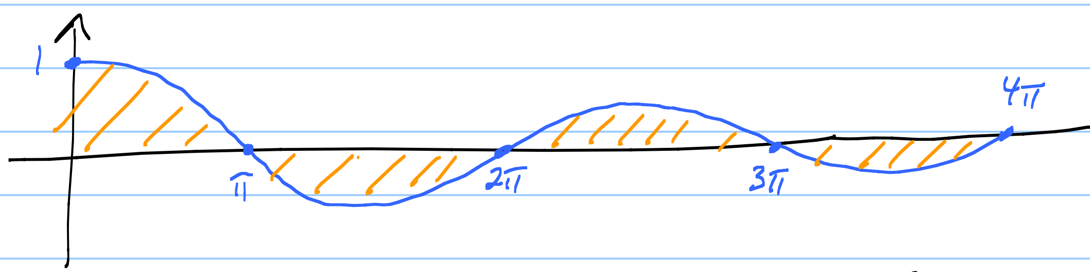
$$\leq 2\int_0^{\pi/2} e^{-\frac{2R}{\pi}t}\,dt$$

$$2\left[\frac{1}{\left(-\frac{2R}{\pi}\right)} e^{-\frac{2R}{\pi}t}\right]_0^{\pi/2}$$

$$= \frac{\pi}{R}\left(1-e^{-R}\right) < \frac{\pi}{R}$$

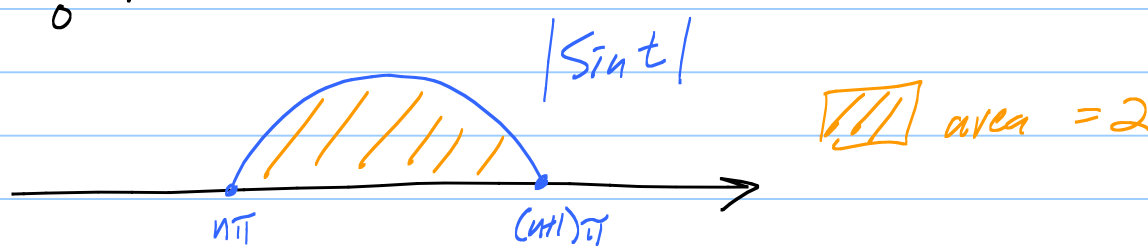
$$\to 0 \text{ as } R \to \infty.$$

$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$ is famous



Alternating series test shows that $\int_0^R$ converges as $R \to \infty$

but $\int_0^R \left|\frac{\sin t}{t}\right| dt \to \infty$. Conditionally convergent.



$$\int_{n\pi}^{(n+1)\pi}: \quad \frac{|\sin t|}{(n+1)\pi} \leq \frac{|\sin t|}{t} \leq \frac{|\sin t|}{n\pi} \qquad [n\pi, (n+1)\pi]$$

$$\frac{2}{(n+1)\pi} \leq \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{t}\,dt \leq \frac{2}{n\pi}$$

Shows that $\int_0^{(N+1)\pi}$ blows up like

harmonic series $\sum_1^N \frac{1}{n} \sim \ln N$

Thm: P, Q polynomials

1) $\deg Q \geq \deg P + 1$ ← instead of $+2$

2) Q has no zeroes on $\mathbb{R}$

Then

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} e^{ix}\, dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P(z)}{Q(z)} e^{iz}$$

where Z_Q^+ = zeroes of Q in UHP.

Pf: Basic est fails us, but Jordan's lemma saves the day.

EX: $I = \int_0^{2\pi} \sin^4\theta \, d\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}$$

Aha! $= \dfrac{z(\theta) - \frac{1}{z(\theta)}}{2i}$ where $z(\theta) = e^{i\theta}$ $\quad C$

$z'(\theta) = ie^{i\theta}$

$$I = \int_0^{2\pi} \left(\frac{z(\theta) - \frac{1}{z(\theta)}}{2i}\right)^4 \frac{ie^{i\theta}\, dt}{\underbrace{ie^{i\theta}}_{iz(\theta)}} \quad dz$$

$$= \int_C \left(\frac{z - \frac{1}{z}}{2i}\right)^4 \cdot \frac{1}{iz}\, dz$$

residue term!

$$= \frac{1}{16}\int_C \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\cdot \frac{1}{iz}\, dz$$

$$= \frac{1}{16}\cdot 2\pi i \,\mathrm{Res}_0\left(\text{eeee}\right) = \frac{1}{16}\cdot 2\pi i\cdot\frac{6}{i} = \frac{3\pi}{4}$$

<u>Method</u>: $R(z,w)$ rational

no blowing up

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

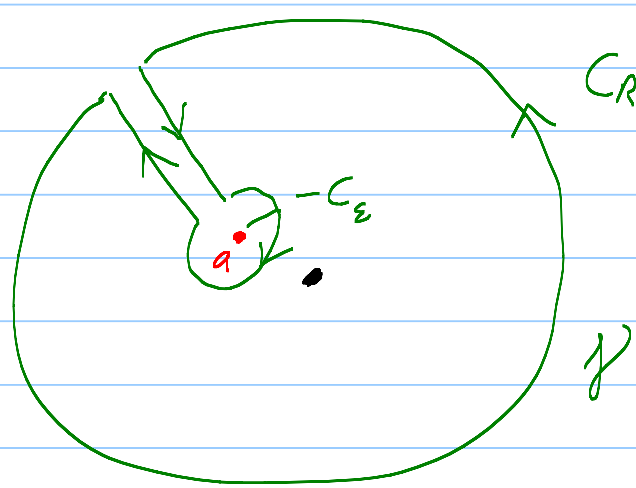
no poles on C

$$= \int_C \underbrace{R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{1}{iz}}_{H(z)}\, dz$$

$$= 2\pi i \sum_{\substack{a \in \text{poles of} \\ H \text{ inside } C}} \mathrm{Res}_a H$$

<u>Stein</u> Keyhole trick

$$0 = \int_\gamma \frac{f(z)}{z-a}\, dz$$



Push parallel lines together.

They cancel.

Get $\left(\int_{-C_\varepsilon} + \int_{C_R} \right) \frac{f(z)}{z-a} \, dz = 0$

$$\int_{-C_\varepsilon} \frac{f(z)}{z-a} \, dz = - \int_{C_\varepsilon} \frac{f(z)}{z-a} \, dz = - \int_0^{2\pi} \frac{f(a+\varepsilon e^{it})}{(a+\varepsilon e^{it}) - a} \, i\varepsilon e^{it} \, dt$$

$$= -i \int_0^{2\pi} f(a+\varepsilon e^{it}) \, dt$$

$\longrightarrow -i \, 2\pi \, f(a)$ because

analytic f is continuous at a.

Cauchy thm $\Rightarrow$ Cauchy integral formula