Lecture 21 Why the "argument" principle

HWK 5 due Thurs, Midterm exam Friday after spring break

f analytic on $D_R(z_0)$, nonvanishing on $C_r(z_0)$, $0<r<R$.

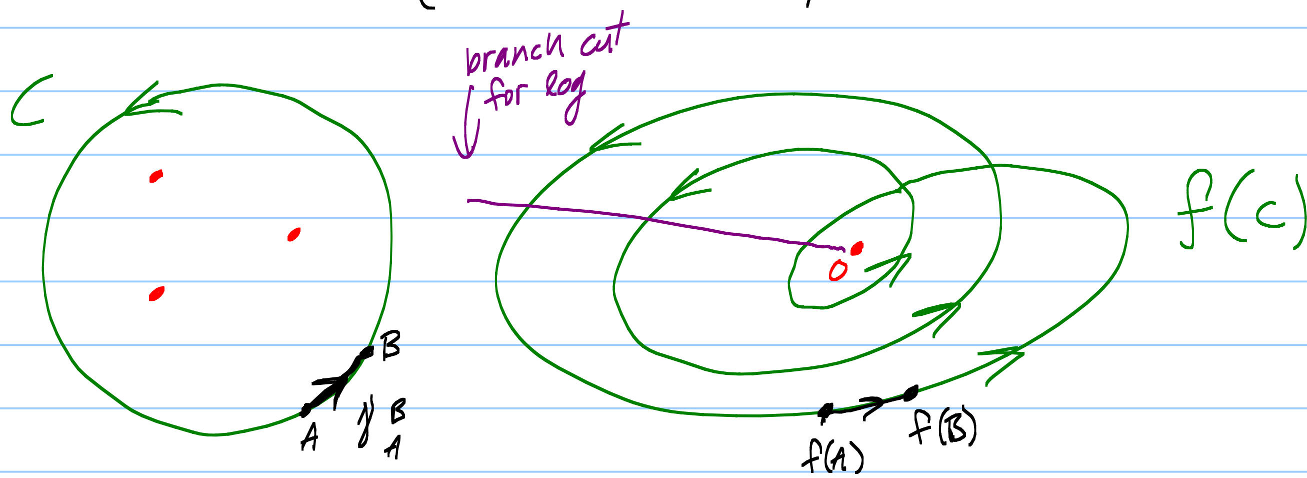
$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \begin{pmatrix} \text{\# zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with multiplicity} \end{pmatrix}$$

If f allowed to have finitely many poles in $C_r(z_0)$ too, then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \begin{pmatrix} \text{\# zeroes } f \\ \text{in } C_r, \\ \text{w/ mult.} \end{pmatrix} - \begin{pmatrix} \text{\# poles } f \\ \text{in } C_r, \\ \text{w/ order} \end{pmatrix}$$

Pf:

$$\mathrm{Res}_a \frac{f'}{f} = \begin{cases} N & a \text{ is zero of order } N \\ -N & a \text{ is pole of order } N \end{cases}$$



$$\int_{\gamma_A^B} \frac{f'}{f}\,dz = \int_{\gamma_A^B} \frac{d}{dz}\Big[\log f(z)\Big]\,dz$$

$$= \log f(B) - \log f(A)$$

$$= \Big(\mathrm{Ln}|f(B)| + i\arg f(B)\Big) - \Big(\mathrm{Ln}|f(A)| + i\arg f(A)\Big)$$

$$= \left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)| \right) + i \underbrace{\Delta_{\gamma_A^B} \arg f}_{\text{does not depend on choice of branch!}}$$

Add up $\int_{\gamma_{A_i}^{A_{i+1}}}$ going around C:

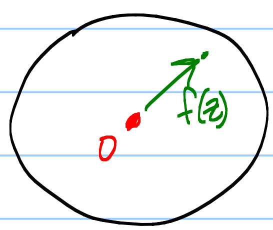
$\operatorname{Ln}|f|$ terms pairwise cancel

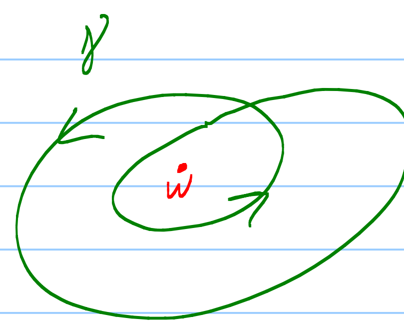
and first one cancels last one!

Aha! $\displaystyle\int_C \frac{f'}{f}\,dz = i \sum \Delta_{\gamma_{A_i}^{A_{i+1}}} \arg f \overset{\text{def}^n}{=} i \,\Delta_C \arg f$

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \frac{1}{2\pi}\Delta_C \arg f = \left(\begin{array}{c}\#\text{ times } f(z)\\ \text{goes around}\\ \text{the origin}\end{array}\right)$$

Arg watch:





<u>Def^n</u>: $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}\,dz$

$$= \left(\begin{array}{c}\text{winding } \#\\ \text{of } \gamma \text{ about } w\end{array}\right) = \operatorname{Ind}_\gamma(w) \leftarrow \text{Index of } w \text{ with respect to } \gamma$$

$$= \left(\begin{array}{l}\#\text{ times } \gamma\\ \text{wraps around}\\ w,\ \text{counterclockwise}\end{array}\right)$$

Note: Argument integral idea with $f(z) = z - w$

Fact: $C : z(t),\ a \le t \le b$

$f(C) : f(z(t)),\ a \le t \le b$

$$\underbrace{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w}\,dz}_{H(w)} = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))}{\underbrace{f(z(t))}_{\zeta}-w}\, z'(t)\,dt$$

$\frac{d}{dt} f(z(t))\, dt$

$$= \frac{1}{2\pi i}\int_{f(C)} \frac{1}{\zeta - w}\, d\zeta$$

$$= \mathrm{Ind}_{f(C)}(w)$$

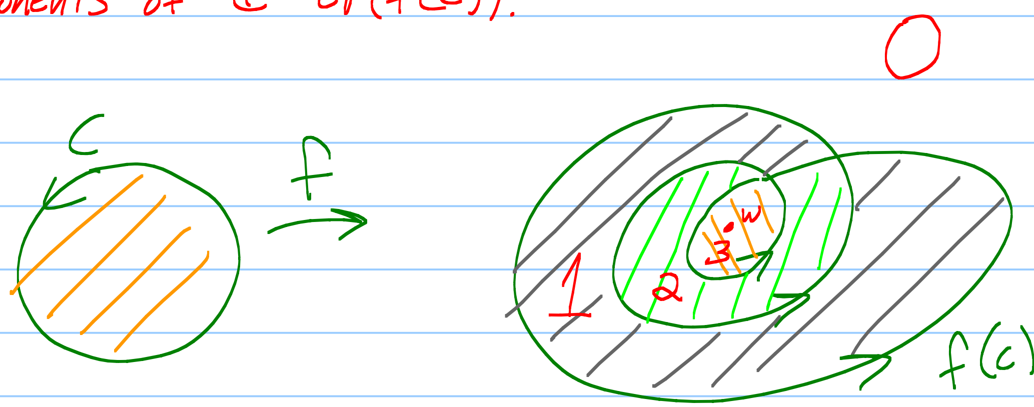
Last time

$H'(w) \equiv 0$,

So H is constant

on connected components of $\mathbb{C} - tr(f(C))$.

Picture:

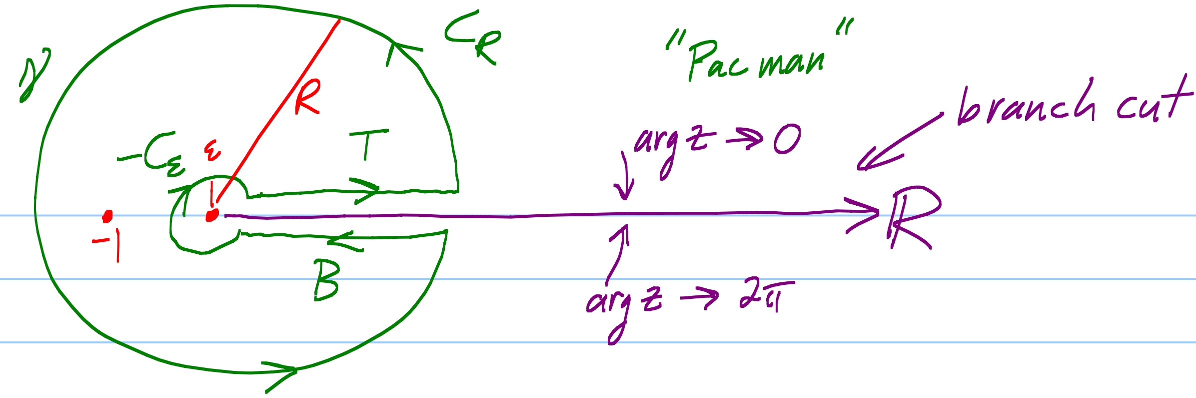


Think about z^3.

EX: $I = \int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx, \quad 0 < \alpha < 1$

$x^{-\alpha} = e^{-\alpha \operatorname{Ln} x}, \quad (x > 0)$

$z^{-\alpha} = e^{-\alpha \log z} \leftarrow$ branch ?



Res Thm
$$\int_\gamma \underbrace{\frac{e^{-\alpha \log z}}{z+1}}_{f(z)}\, dz = 2\pi i \operatorname{Res}_{-1} \frac{e^{-\alpha \log z}}{z+1}$$
$$= 2\pi i \frac{e^{-\alpha \log(-1)}}{(1)}$$
$$= 2\pi i \; e^{-\alpha(\ln|-1| + i\pi)}$$
$$= 2\pi i \; e^{-i\alpha\pi}$$

Limits as "mouth" closes:
$$\int_T f(z)\, dz = \int_\varepsilon^R \frac{e^{-\alpha(\ln t + i\cdot 0)}}{t+1}\, dt \quad \leftarrow \text{want!}$$
$$\int_B f(z)\, dz = -\int_\varepsilon^R \frac{e^{-\alpha(\ln t + i\cdot 2\pi)}}{t+1}\, dt$$
$$= -\underbrace{e^{-i\alpha 2\pi}}_{\text{not} = -1.} \underbrace{\int_\varepsilon^R \frac{e^{-\alpha \ln t}}{t+1}\, dt}_{\text{want}}$$

On C_R : $\left|z^{-\alpha}\right| = |z|^{-\alpha}$ (claim)

$$\left|e^{-\alpha \log z}\right| = \left|e^{-\alpha \log Re^{it}}\right| = \left|e^{-\alpha(\ln R + it)}\right|$$

$$= \underbrace{\left|e^{-\alpha \ln R}\right|}_{R^{-\alpha}} \underbrace{\left|e^{-\alpha it}\right|}_{1} = R^{-\alpha} \checkmark$$

$$|z+1| \geq \big||z|-|1|\big| = R-1 \quad \text{if} \quad R>1$$

$$\left|\int_{C_R} f\, dz\right| \leq \left(\frac{R^{-\alpha}}{R-1}\right)\cdot(2\pi R) \qquad \text{if} \quad R>1$$

$$\longrightarrow 0 \quad \text{as} \quad R\to\infty \quad \text{because} \quad \underline{\alpha > 0}.$$

<u>On C_ε</u> : $|z+1| \geq \big||z|-|1|\big| \geq 1-\varepsilon$ if $|z| = \varepsilon < 1$

$$\left|\int_{C_\varepsilon} f\, dz\right| \leq \frac{\varepsilon^{-\alpha}}{1-\varepsilon}\cdot 2\pi\varepsilon \qquad \text{if} \quad 0<\varepsilon<1$$

$$\longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{because} \quad \underline{\alpha < 1}.$$

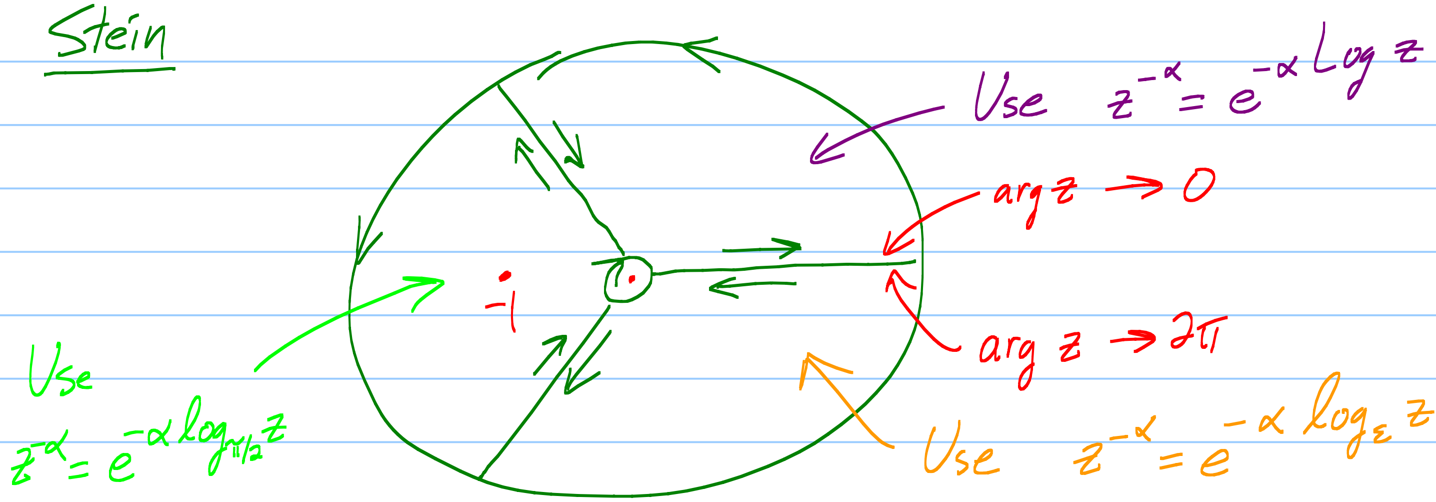
Close mouth, let $\varepsilon \to 0$, let $R \to \infty$. Get

$$\Bigg(\underbrace{\int_T + \int_B}_{\left(1-e^{-i\alpha 2\pi}\right) I} + \underbrace{\int_{-C_\varepsilon}}_{\downarrow\, 0} + \underbrace{\int_{C_R}}_{\downarrow\, 0}\Bigg) f\, dz = 2\pi i\, e^{-i\alpha\pi}$$

So $$I = \frac{2\pi i \, e^{-i\alpha\pi}}{1 - e^{-i\alpha 2\pi}} \cdot \frac{e^{i\alpha\pi}}{e^{i\alpha\pi}}$$

$$= \pi \left(\frac{2i}{e^{i\alpha\pi} - e^{-i\alpha\pi}} \right) = \frac{\pi}{\sin \alpha\pi}$$

Stein



Add up 3 $\int$'s using different branches of log to cancel out first two cuts. Third cuts don't cancel!