Lecture 23 Linear fractional transformations (LFTs)

HWK 6 due Thurs, 3/10

$$L(z) = \frac{az+b}{cz+d} = w \qquad D = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$$

Solve for z:

($D=0$ means rows lin dependent $\Rightarrow$ num $L(z) = (\text{const})$ denom $L(z)$ or worse: denom $\equiv 0$.)

$$z = \frac{dw-b}{-cw+a} = L^{-1}(w)$$

$$\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \det\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbb{A}} = D \neq 0$$

$$\mathbb{A}^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Cool thing: LFT's form a group under composition.

$$L \sim \mathbb{A} \qquad id(z) = z = \frac{1\cdot z+0}{0\cdot z+1} \sim \mathbb{I}$$

$$L^{-1} \sim \mathbb{A}^{-1}$$

$$\left.\begin{array}{l} L_1 \sim \mathbb{A}_1 \\ L_2 \sim A_2 \end{array}\right\} \quad L_1 \circ L_2 \sim \mathbb{A}_1 \mathbb{A}_2 \leftarrow \text{matrix mult}$$

Fact LFTs isomorphic to grp of 2x2 complex matrices with $\det \neq 0$.

Fact LFTs: $\left\{\begin{array}{l}\text{lines,} \\ \text{circles}\end{array}\right\} \circlearrowleft$

Lemma: LFTs are generated by 4 basic maps

1) $z \mapsto rz$, $r \in \mathbb{R}^+$ (stretch)

2) $z \mapsto e^{i\theta} z$ (rotation)

3) $z \mapsto z + b$ (translation)

4) $z \mapsto \frac{1}{z}$ (inversion)

<u>Pf</u>: Case $c = 0$: $L(z) = \underbrace{A}_{re^{i\theta}} z + B$ 1, 2, 3 all we need.

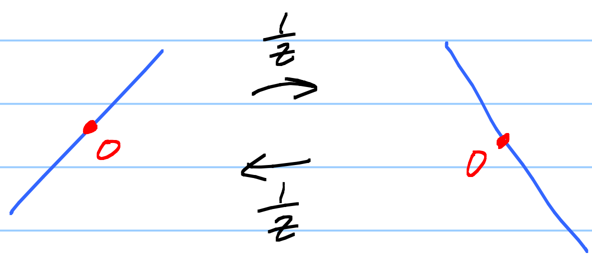
<u>Case $c \neq 0$</u>: $$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\overbrace{\left(b - \frac{ad}{c}\right)}^{Re^{i\psi}}}{\underbrace{c}_{c = re^{i\theta}}z + d}$$

Need 7 basic maps to get L!

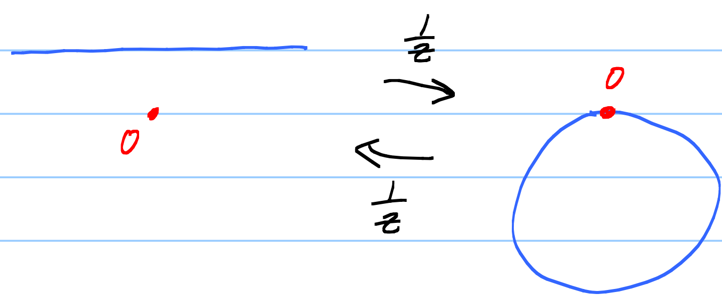
<u>Cor</u>: LFTs : $\left\{\begin{matrix}\text{lines,} \\ \text{circles}\end{matrix}\right\} \circlearrowleft$

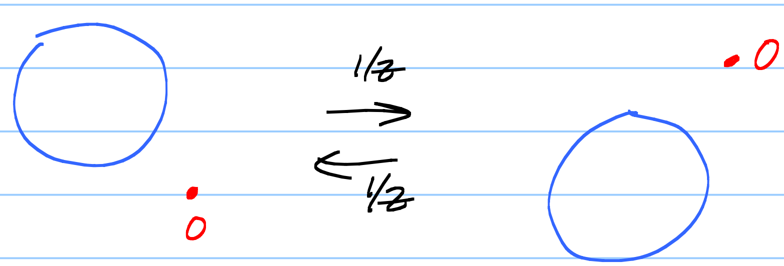
Pf: 1, 2, 3 : $\{\text{lines}\} \circlearrowleft$, $\{\text{circles}\} \circlearrowleft$,

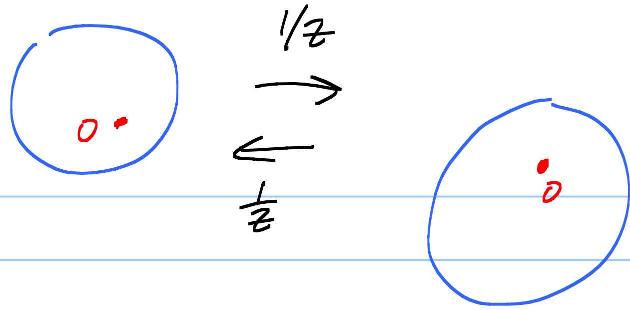
4 mixes them up:



$0 \mapsto \infty$
$\infty \mapsto 0$

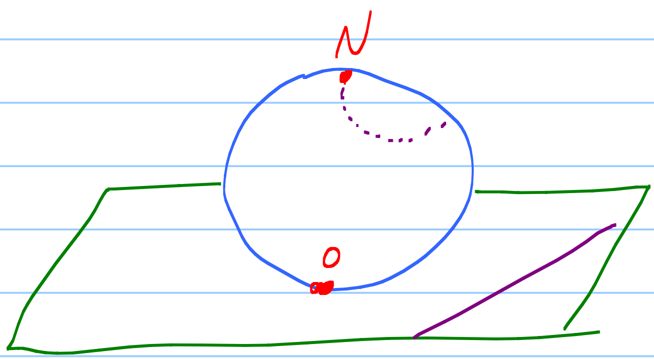






Pf: HS analytic geom.

Fun fact: On Riemann sphere $\{\text{lines, circles}\} \sim$ slices



LFTs : $\hat{\mathbb{C}} \circlearrowleft$ 1-1 onto

$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c} \qquad (c \neq 0)$$

$$\begin{cases} L(\infty) = \frac{a}{c} \\ L\left(\frac{-d}{c}\right) = \infty \end{cases}$$

Lemma: An LFT that fixes 3 distinct pts must be the identity.

Pf: $\quad L(z_j) = z_j \qquad j = 1, 2, 3$

$$\frac{az_j + b}{cz_j + d} = z_j$$

$$cz_j^2 + (d-a)z_j - b = 0$$

Quad eqn with 3 roots! All coeff must $= 0$.

So $\quad c = 0, \quad \underbrace{d - a = 0}_{a=d}, \quad b = 0$

$D = ad \neq 0, \qquad$ So $\; a \neq 0, d \neq 0.$

$$L(z) = \frac{az + 0}{0 \cdot z + d} = \Big(\underbrace{\frac{a}{d}}_{1}\Big) z$$

Cor: If two LFTs agree at 3 pts, they are the same.

3 pts determine circle

2 pts (plus the pt at ∞) determine a line.

Useful LFTs

$$\frac{z-a}{z-b} \qquad a \mapsto 0,\quad b \mapsto \infty$$

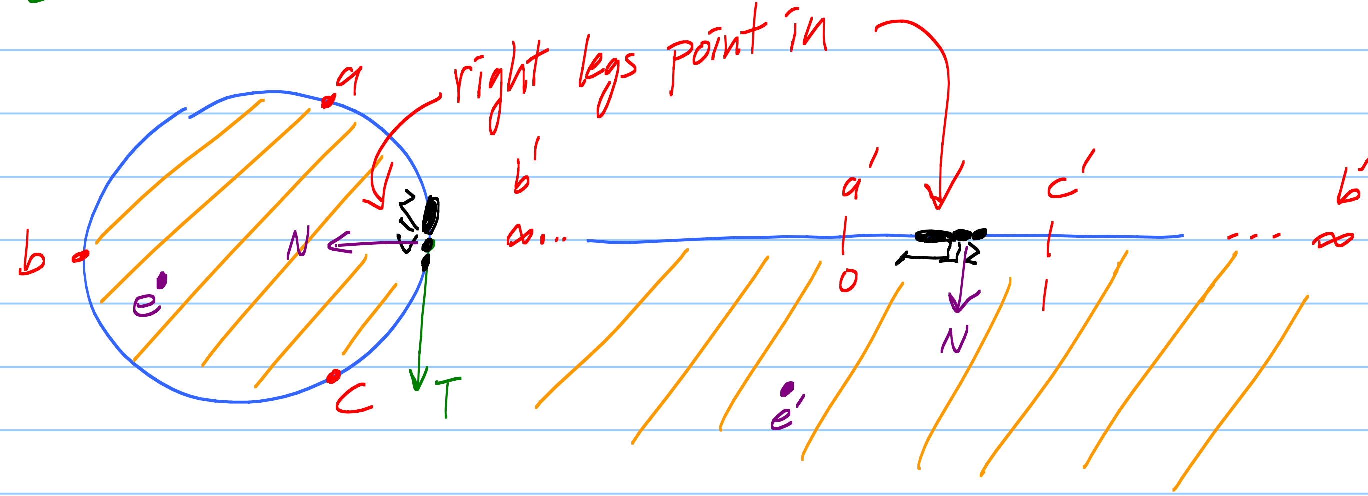
$$\left(\frac{c-b}{c-a}\right)\cdot\frac{z-a}{z-b} \qquad a \mapsto 0,\quad b \mapsto \infty,\quad c \mapsto 1$$

Prob: Find an LFT: $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$ in finite $\mathbb{C}$

$$\begin{array}{ccccc} & L_1 & & L_2 & \\ z_1 & \mapsto & 0 & \leftarrow & w_1 \\ z_2 & \mapsto & \infty & \leftarrow & w_2 \\ z_3 & \mapsto & 1 & \leftarrow & w_3 \end{array}$$

$L_2^{-1} \circ L_1$ solves prob.

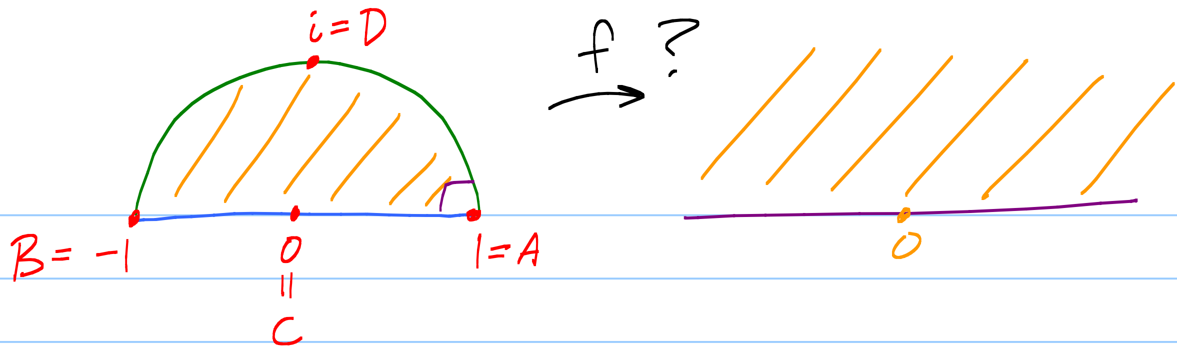
EX:



Bug test: Use conformality to see which side goes to what side.

Check one pt test:

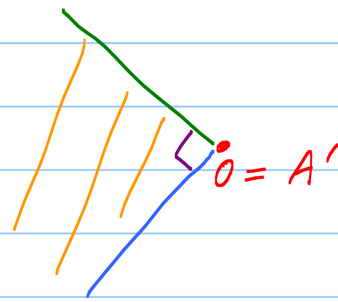
EX:



Trick: blow a corner to ∞.

$$L(z) = \frac{z-1}{z-(-1)}$$

Hmmm.



Quadrant!

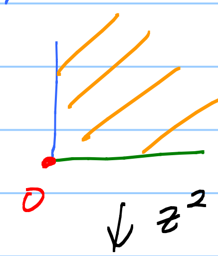
$L(1) = 0$

$L(-1) = \infty$

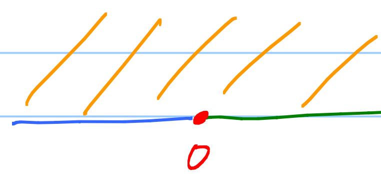
$L(i) = \frac{i-1}{i+1} = i$

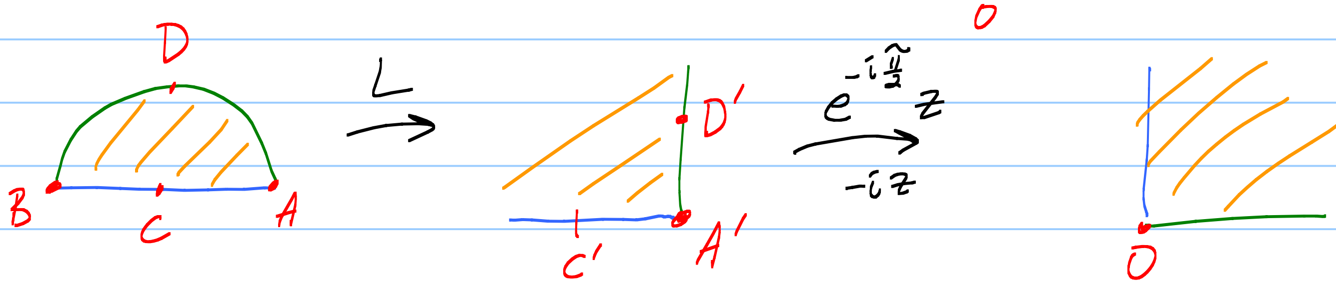
$L(0) = \frac{0-1}{0+1} = -1$

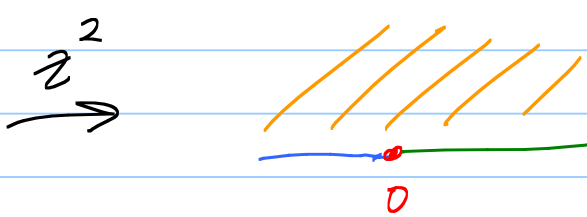
rotate



Square







$$f(z) = \left[-i\left(\frac{z-1}{z+1}\right)\right]^2 = -\left(\frac{z-1}{z+1}\right)^2$$

Big picture