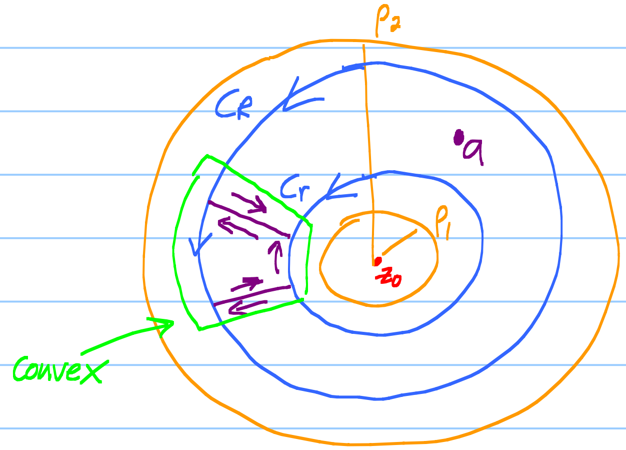
Lecture 26 Laurent expansions

HWK 6 due tonight
at 11:59 in GS
Midterm exam Fri, March 25

Find Practice problems for the midterm on the home page (not to be turned in). Feel free to discuss them on Piazza.

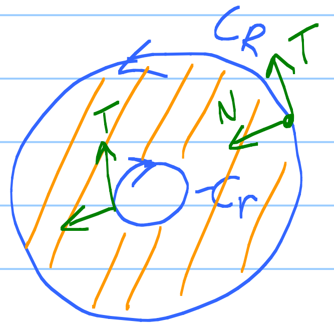


Suppose f is analytic on $A(\rho_1, \rho_2) = \{z : \rho_1 < |z - z_0| < \rho_2\}$.

$$\rho_1 < r < R < \rho_2$$

Cauchy's theorem for an annulus: $\left(\int_{C_R} - \int_{C_r}\right) f\, dz = 0$

Pf: Add up "pieces of pie". Use Cauchy on convex. Cancel cuts.



$\overset{\text{CCL}}{C_R} \cup \underbrace{(-C_r)}_{\text{CL}}$ "standard orientation of the boundary"

(Inward pointing normal vector is to the left of the T vector.)

Cauchy integral formula: $f(a) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{z-a}\, dz$

Pf: Apply the C. Thm to $F(z) = \frac{f(z) - f(a)}{z - a}$,

which has a removable sing at $z = a$.

$$0 = \left(\int_{C_R} - \int_{C_r} \right) \frac{f(z)}{z-a} - \frac{f(a)}{z-a} \, dz$$

Aha! $\left(\int_{C_R} - \int_{C_r} \right) \frac{f(a)}{z-a} \, dz = f(a) \left[\underbrace{\int_{C_R} \frac{1}{z-a} dz}_{2\pi i} - \underbrace{\int_{C_r} \frac{1}{z-a} dz}_{=0} \right]$

a inside C_R, outside C_r.

Laurent series :

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\substack{\text{converging in } D_{\rho_2}(z_0), \\ \text{absolutely on } D_R(z_0).}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}}_{\substack{\text{converging in} \\ \{z : |z-z_0| > \rho_1\}, \\ \text{unif in } \{z : |z-z_0| > r\}}}$$

$$a_n = \frac{1}{2\pi i} \int_{C_{\tilde{r}}} \frac{f(z)}{(z-z_0)^{n+1}} \, dz \quad \leftarrow n \in \mathbb{Z}$$

$\rho_1 < \tilde{r} < \rho_2$

Cauchy thm $\Rightarrow$ indep of choice of $\tilde{r}$

Theorem: f has an isolated singularity at z_0.

1) $a_{-n} = 0$ for $n = 1, 2, 3, \ldots \iff z_0$ is removable

2) Only finitely many $a_{-n} \neq 0$ $(n \in \mathbb{N})$

$\iff z_0$ is a pole

3) Infinitely many $a_{-n} \neq 0$ $(n \in \mathbb{N}) \iff z_0$ essential

Pf of expansion: Let $z_0 = 0$.

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{\underbrace{w-z}_{w(1-\frac{z}{w})}} dw - \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{\underbrace{w-z}_{-z(1-\frac{w}{z})}} dw$$

$|\frac{z}{w}| < 1$ $\qquad$ $|\frac{w}{z}| < 1$

$$= \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\left(1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \cdots \left(\frac{z}{w}\right)^N\right) dw$$

+ remander term $\longrightarrow 0$ unif on a shrunk disc.

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w^{n+1}} dw\right)}_{a_n} z^n$$

$$-\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{\underbrace{w-z}_{-z(1-\frac{w}{z})}} dw = \frac{1}{2\pi i}\frac{1}{z}\int_{C_r} \frac{f(w)}{1-\frac{w}{z}} dw$$

$|\frac{w}{z}| < 1$

$$= \frac{1}{2\pi i}\cdot\frac{1}{z}\int_{C_r} f(w)\left[1 + \left(\frac{w}{z}\right) + \cdots + \left(\frac{w}{z}\right)^N + \underbrace{\frac{\left(\frac{w}{z}\right)^{N+1}}{1-\frac{w}{z}}}_{\varepsilon_N(\frac{w}{z})}\right] dw$$

$$= \sum_{n=1}^{N+1} \underbrace{\left(\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w^{-n+1}} dw\right)}_{a_{-n}} \frac{1}{z^n} + \mathcal{E}_N(z)$$

Finally $$\left|\mathcal{E}_N(z)\right| < \frac{1}{2\pi}\,\frac{1}{|z|}\cdot \operatorname*{Max}_{C_r}|f|\ \frac{\left(\frac{r}{|z|}\right)^{N+1}}{1-\frac{r}{|z|}}$$

$$\to 0 \quad \text{as} \quad N\to\infty.$$

Restrict z to smaller annulus to get unif estimate.

Notation: $\sum_{-\infty}^{\infty} a_n z^n$ converges means that

both $\sum_{0}^{\infty}$ and $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ converge.

Remark: At an isolated sing, Laurent exp is valid on punctured disc.

Pf of Isolated sing thm:

(1): $(\Longrightarrow)$ If $a_{-n}=0\ \forall\, n\in\mathbb{N}$, then

$f=$ power series on $D_R(z_0)-\{z_0\}$.

So z_0 removable.

$(\Longleftarrow)$ z_0 removable. $(n\in\mathbb{N})$

$$a_{-n} = \frac{1}{2\pi i}\int_{C_{\tilde r}} \frac{f(w)}{(w-z_0)^{-n+1}}\,dw$$

$$= \frac{1}{2\pi i}\int_{C_{\tilde r}} \underbrace{f(w)\,(w-z_0)^{n-1}}_{\text{analytic inside}}\,dw = 0$$

Cauchy

✓

(2) Lemma: Laurent expansions are unique.

Why:
$$a_N = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{N+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_r} \left[\sum_{-\infty}^{\infty} b_n (z-z_0)^n\right] \frac{1}{(z-z_0)^{N+1}}\,dz$$

$$= \sum_{-\infty}^{\infty} b_n \underbrace{\left(\frac{1}{2\pi i}\int_{C_r} (z-z_0)^{n-N-1}\,dz\right)}_{\begin{cases} =0 & \underbrace{n-N-1 \neq -1}_{n \neq N} \\ =1 & n=N \end{cases}}$$

$$= b_N \quad \checkmark$$

(2): ($\Rightarrow$) $f(z) = (\text{Princ Part}) + \text{power series}$.

So z_0 is a pole.

($\Leftarrow$) Princ part + power series is a Laurent exp. It is the unique Laurent exp.

(3): Left over case!

Def^n: At an isolated sing z_0

$$\mathrm{Res}_{z_0} f = a_{-1} = \frac{1}{2\pi i} \int_{C_r} f(w)\, dw$$

Meaning of residue: f has an analytic antiderivative on $D_R(z_0) - \{z_0\} \iff \mathrm{Res}_{z_0} f = 0$.

Think: Residue term leads to log terms.