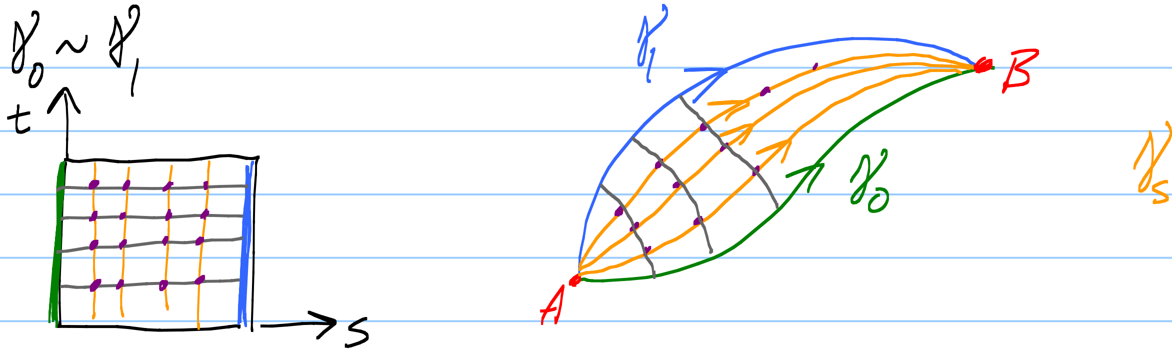
Lecture 28 Homotopy, simply connected domains, the Cauchy Theorem



Homotopy : $H : [0,1] \times [0,1] \longrightarrow \Omega$ continuous

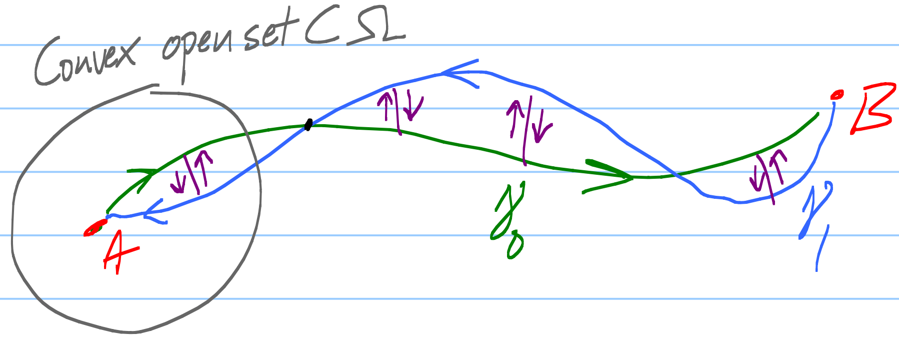
$$\begin{cases} H(0,t) = z_0(t) & : \gamma_0 \\ H(1,t) = z_1(t) & : \gamma_1 \end{cases}$$

$$\begin{cases} H(s,0) = A \\ H(s,1) = B \end{cases}$$

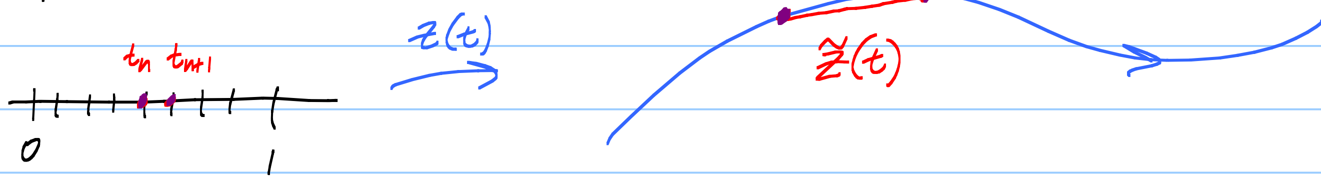
Think $\gamma_s : z_s(t) = H(s,t)$

Feeling: If γ_0 is "close" to γ_1 in Ω, then

$$\int_{\gamma_0} f\, dz = \int_{\gamma_1} f\, dz \qquad (f \text{ analytic on } \Omega)$$

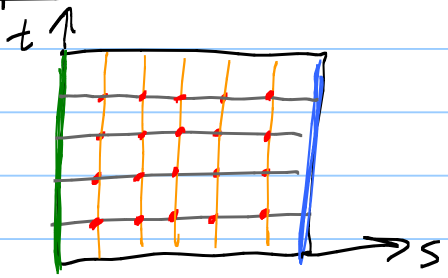


Step: Piecewise linearize a curve :

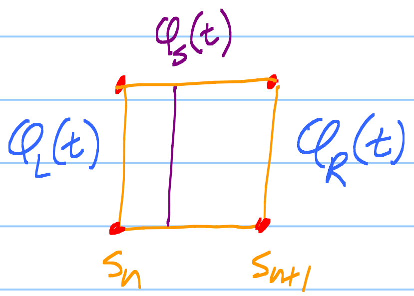


$$\tilde{z}(t) = z(t_n) + \frac{t-t_n}{t_{n+1}-t_n}\left[z(t_{n+1}) - z(t_n)\right]$$

Step 2: Do this in the vertical dir for homotopy.



Step 3: On little square



$$\varphi_s(t) = \varphi_L(t) + \frac{s-s_n}{s_{n+1}-s_n}\left[\varphi_R(t) - \varphi_L(t)\right]$$

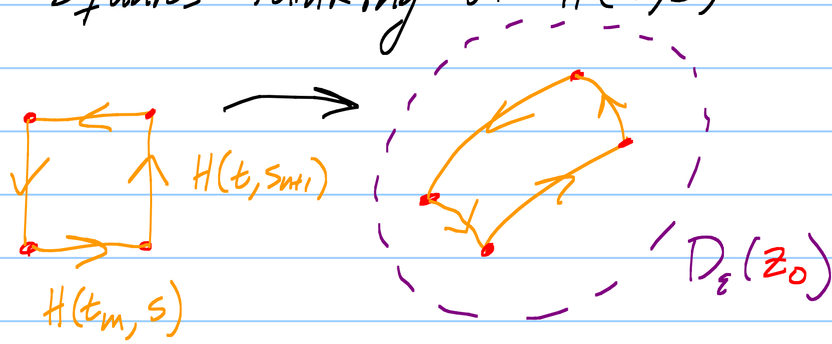
Check Get piecewise linear cont fcn on $[0,1]\times[0,1]$.

New homotopy, $H(t,s)$. Note: $\begin{cases} H(0,s) = A \\ H(1,s) = B \end{cases}$ ✓

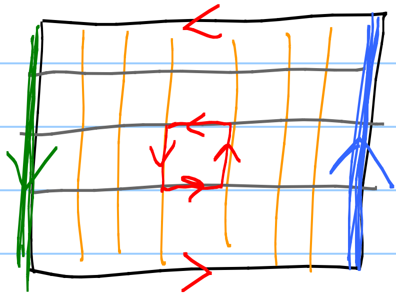
Thm: f analytic on Ω, $\gamma_0 \sim \gamma_1$ in Ω.

Then $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$. (Cauchy's Thm)

Pf: Add up $\int$ around little squares thinking of $H(t,s)$ as parametrizing curves:



Add up squares:



Use Cauchy on convex. ε comes from unif cont of cont fcn H on compact $[0,1]\times[0,1]$.

Cancel all the inner sides.

$$\int_{\text{Bottom}} = \int f(A)\underbrace{\left[\frac{d}{ds}A\right]}_{\equiv 0} ds = 0.$$ Same for top

Right side: $\int_{\gamma_1} f\, dz.$

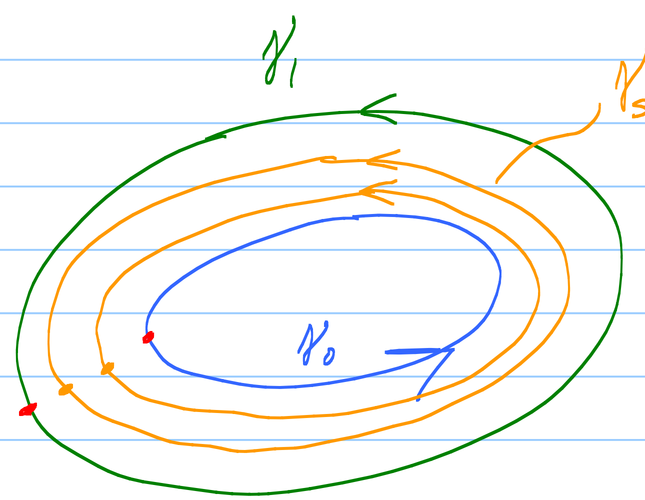
Left side: $-\int_{\gamma_0} f\, dz.$

Get $\int_{\gamma_0} = \int_{\gamma_1}$ ✓

Def^n: Two closed curves γ_0 and γ_1 in Ω are homotopic in Ω if there is a homotopy $H(t,s)$ such that

$$\begin{cases} H(t,0) = z_0(t) & : \gamma_0 \\ H(t,1) = z_1(t) & : \gamma_1 \end{cases}$$

$$H(0,s) = H(1,s)$$

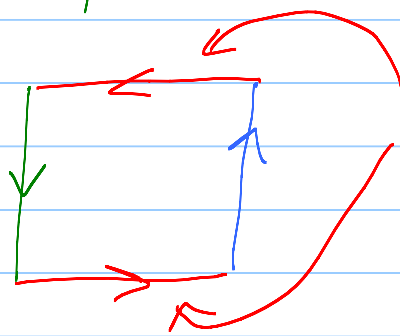


Think: $\gamma_s : z_s(t) = H(t,s).$

Thm: f anal on Ω, $\gamma_0 \sim \gamma_1$ in Ω,

then $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$

Pf: Same pf, but



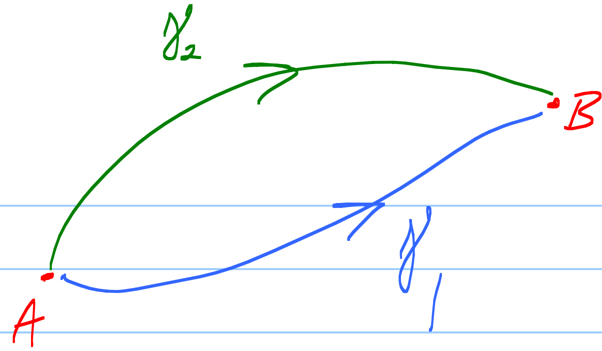
Def^n: A domain in $\mathbb{C}$ is simply connected if every closed curve in Ω is "homotopic to a point" in Ω. $[\,z(t) \equiv z_0 \in \Omega,\ 0 \le t \le 1\,]$

The Cauchy Thm: f analytic on a simply connected domain Ω, then $\int_\gamma f\,dz = 0$ for every closed curve in Ω. Also, for any two curves γ_1 and γ_2 that both start at A and stop at B,

$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz.$$

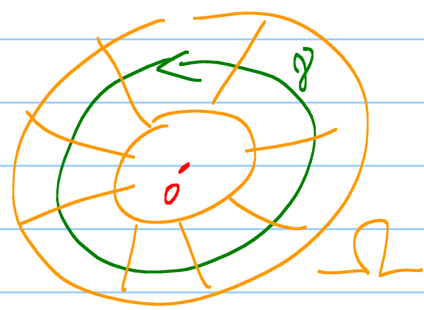
Pf: $\gamma \sim z_0$ $\qquad \int_\gamma f\,dz = \int_0^1 f(z_0) \underbrace{\frac{d}{dt} z_0}_{\equiv 0}\,dt = 0$

Part 2:



Closed curve $\gamma = \gamma_1 \cup -\gamma_2$. Use part 1.

EX:



$$\int_\gamma \frac{1}{z}\,dz = 2\pi i$$ ← Not zero

Holes mess up Cauchy's thm.