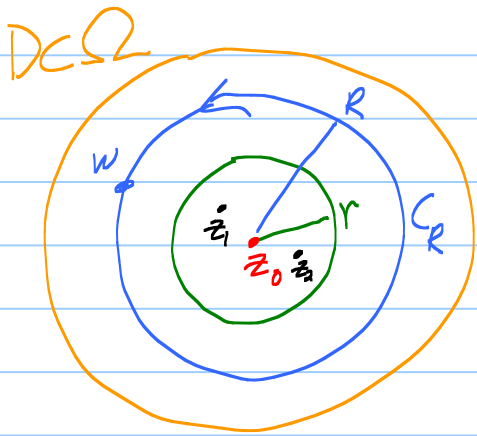
Lecture 30 Montel's theorem

HWK 7 due Tues, 4/12

Arzela-Ascoli Thm: Baby Rudin p. 158
Ahlfors p. 219–227
Stein p. 225



Lemma Uniformly bounded family $\mathcal{F}$ of analytic fcns on a domain are uniformly equicontinuous on compact subsets.

$f \in \mathcal{F}$

$$\left| f(z_1) - f(z_2) \right| = \left| \frac{1}{2\pi i} \int_{C_R} f(w) \underbrace{\left[\frac{1}{w - z_1} - \frac{1}{w - z_2} \right]}_{\frac{z_1 - z_2}{(w-z_1)(w-z_2)}} dw \right|$$

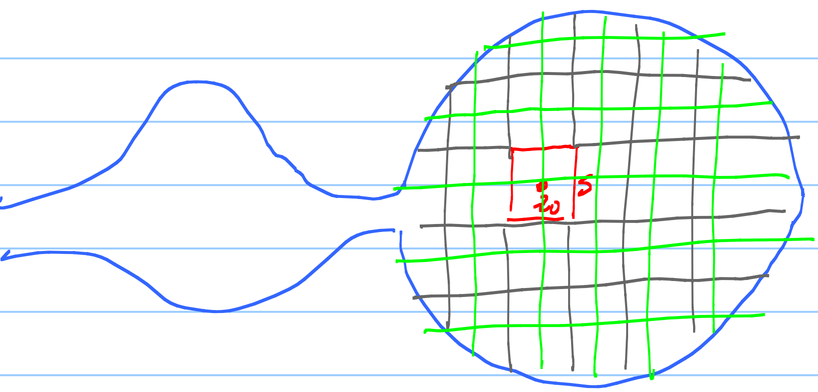
$$\leq |z_1 - z_2| \frac{1}{2\pi} \underbrace{\max_{C_R} |f|}_{\text{indep of } f} \left(\frac{1}{(R-r)^2} \right) (2\pi R)$$

Finally, cover compact K by discs $D_r(z_0)$, $z_0 \in K$.
Take finite subcover.

Montel's thm Suppose f_n are analytic on a domain Ω and are uniformly bounded on compact subsets of Ω. Then there is a subseq f_{n_k} that converges uniformly on compact subets (to analytic f).

Pf: Step 1: "Exhaust" Ω by compact sets:

$$K_1 \subset K_2 \subset K_3 \subset \cdots \subset \Omega, \quad \bigcup_j K_j = \Omega$$



K_1 = Square 1

K_2 = Union of squares side s inside $\Omega \cap D_{10s}(z_0)$

K_3 = Union of squares of side $s/2$ inside $\Omega \cap D_{100s}(z_0)$

$\frac{s}{4}, \frac{s}{8}, \frac{s}{16}, \cdots \qquad 10s, 100s, 1000s, \ldots$

Fancier version: Domains $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega$

$K_1 = \overline{\Omega_1}, \; K_2 = \overline{\Omega_2}, \; \ldots$

$$\mathring{K}_j \subset K_{j+1}$$

Step 2: $z \in \mathbb{C}$; $\operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}$ is a countable dense subset of $\mathbb{C}$.

Fix K_j. Let $\{z_n\}_{n=1}^{\infty}$ be a countable dense subset of K_j from above.

Step 3: $f_1(z_1), f_2(z_1), f_3(z_1), \ldots$ is a bdd seq.

So $\exists$ conv subseq $f_{1,1}, f_{1,2}, f_{1,3} \ldots$

Now $f_{1,1}(z_2), f_{1,2}(z_2), f_{1,3}(z_2)$ is a bdd seq.

So $\exists$ conv subseq $f_{2,1}, f_{2,2}, f_{2,3}, \ldots$

$f_{2,1}(z_3), f_{2,2}(z_3), \ldots$ bdd seq

etc.

Aha! $\underbrace{f_{1,1}}_{F_1}, \underbrace{f_{2,2}}_{F_2}, \underbrace{f_{3,3}}_{F_3}, \ldots$

converges at all pts in dense subset of K_j.

Step 4: F_n are uniformly Cauchy on K_j. $\varepsilon > 0$

$$|F_n(z) - F_m(z)| = |\overbrace{\underbrace{F_n(z) - F_n(z_j)}_{\text{pick } z_j \text{ close to } z.\text{ equicont makes small}}}^{|\,| < \frac{\varepsilon}{3}} + \underbrace{F_n(z_j) - F_m(z_j)}_{F_n(z_j) \text{ conv} \Rightarrow \text{unif Cauchy}} + \overbrace{\underbrace{F_m(z_j) - F_m(z)}}^{|\,| < \frac{\varepsilon}{3}}|$$

Pick $z_0 \in K_j$. Get $D_\delta(z_0)$ from equicont. fact. $\frac{\varepsilon}{3}$, Shrink: $\frac{\delta}{2}$.

Pick z_j in $D_{\delta/2}(z_0)$. Get estimate on $D_{\delta/2}(z_0)$.

Cover K_j by discs. Take finite subcover. ✓

Step 5: F_n unif Cauchy on K_j. So it converges unif.

$\underbrace{F_{1,1}}_{F_1}, \underbrace{F_{1,2}}_{F_2}, \underbrace{F_{1,3}}_{F_3}, \ldots$ conv unif on K_1

Subseq $F_{2,1}, F_{2,2}, F_{2,3}, \ldots$ conv unif on K_2

Subseq $F_{3,1}, F_{3,2}, F_{3,3}, \ldots$ conv unif on K_3

etc.

$F_{n,n}$ is a subseq of f_n converges uniformly on each K_j!

Step 6 Finally, given compact $K \subset \Omega$, there is a K_j with $K \subset K_j$.

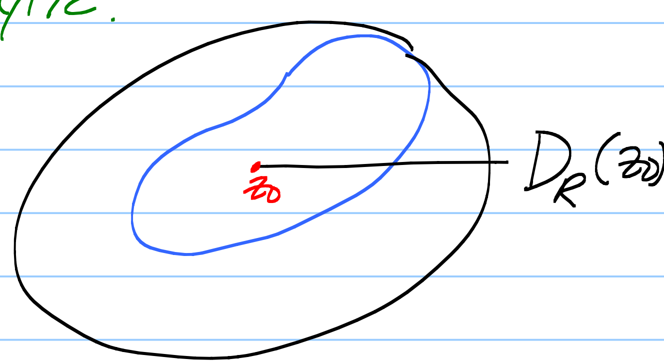
Technical point. $\Omega = \mathbb{C}$. No $f : \mathbb{C} \xrightarrow[\text{onto}]{1-1} D_1(0)$ analytic, Liouville's.

If Ω is simply connected domain and $\Omega \neq \mathbb{C}$,

Riemann mapping thm $\Rightarrow \exists\ f : \Omega \xrightarrow[\text{onto}]{1-1} D_1(0)$ analytic.

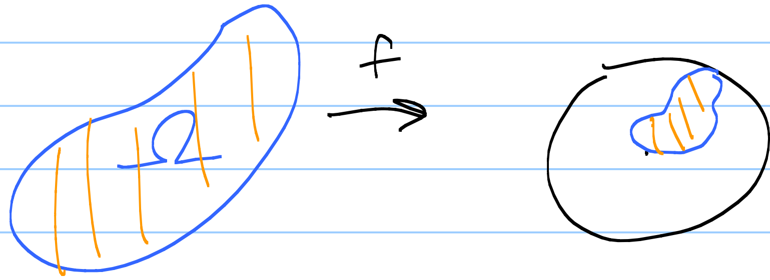
Lemma Given Ω s.c. $\neq \mathbb{C}$, $\exists\ f : \Omega \to D_1(0)$ that is 1-1 and analytic.

Easy if Ω is bounded:



$R = \max_{z \in \overline{\Omega}} |z - z_0|$

$f(z) = \frac{z - z_0}{R}$ maps Ω 1-1 into $D_1(0)$, analytic.



What to do if Ω unbounded?

$\Omega \neq \mathbb{C}$. $\exists\, z_0 \in \mathbb{C} - \Omega$.

Hmmm. $z - z_0$ is nonvanishing on Ω.

$\exists$ analytic $g(z)$ on Ω with $g(z)^2 = z - z_0$

Show g is 1-1, and misses a disc.