Lecture 31 Proof of the Riemann mapping theorem

HWK 7 due Thurs, 4/7

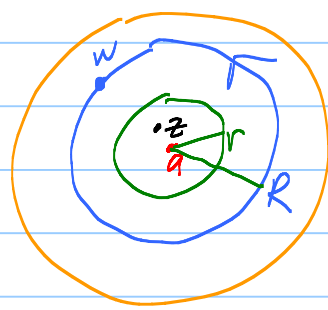
Thm: Ω simply connected domain $\Omega \neq \mathbb{C}$. There is an analytic (conformal) map $f:\Omega \to D_1(0)$ that is one-to-one and onto.

Lemma: If f_n analytic on Ω and $f_n \twoheadrightarrow f$, then $f_n^{(k)} \twoheadrightarrow f^{(k)}$ for each k.

Way false $\mathbb{R} \to \mathbb{R}$: $h_N(t) = \sum_{n=1}^{N} \frac{1}{2^n} \sin(2^n t) \leftarrow h_N \to$ nowhere diff fcn

Montel's false $\mathbb{R} \to \mathbb{R}$: $\sin(2^n t)$

Pf of lemma



$D \subset \Omega$

$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| = \left| \frac{k!}{2\pi i} \int_{C_R} \frac{f_n(w) - f(w)}{(w-z)^{k+1}}\, dw \right|$$

$$\leq \frac{k!}{2\pi} \frac{\max_{C_R} |f_n - f|}{|R-r|^{k+1}} (2\pi R) \to 0$$

See unif conv on $D_r(a)$. Given $K \subset\subset \Omega$, cover by $D_r(z_0)$.

Lemma: $\exists$ $f : \Omega \neq \mathbb{C}$, s.c. $\longrightarrow D_1(0)$ that is 1-1.

Pf: Easy if Ω bounded: $f(z) = \dfrac{z}{R}$ $\quad R = \max\limits_{z \in \Omega} |z|$.

Unbounded: $\Omega \neq \mathbb{C}$. So $\exists$ $w_0 \in \mathbb{C} - \Omega$.

Now $z - w_0$ is nonvanishing on Ω s.c.

So $\exists$ analytic $g(z)$ such that $g(z)^2 = z - w_0$.

Claim: g is 1-1:

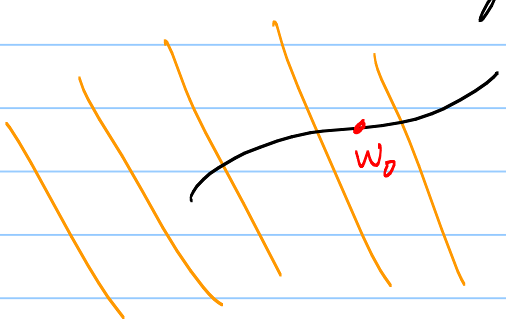
$$g(z_1) = g(z_2)$$
$$g(z_1)^2 = g(z_2)^2$$
$$z_1 - w_0 = z_2 - w_0 \quad \checkmark$$

Claim: $g(\Omega)$ misses a disc:

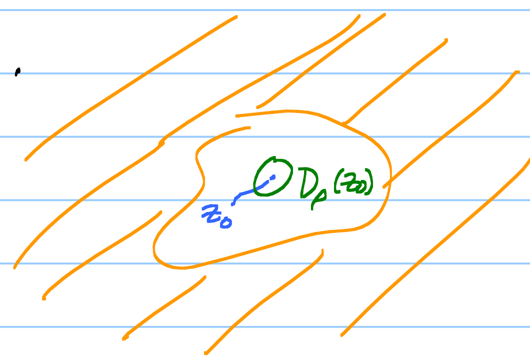
g not constant. Open mapping thm $\Rightarrow$ $\exists$ disc $D_\rho(z_0) \subset g(\Omega)$.

Aha! $-D_\rho(z_0) = \{-z : z \in D_\rho(z_0)\} \cap g(\Omega) = \emptyset$

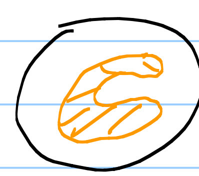
because $g(z)^2$ is 1-1.



$\xrightarrow[1-1]{g}$



$\xrightarrow[1-1]{} \dfrac{\rho}{z - z_0}$



$D_1(0)$

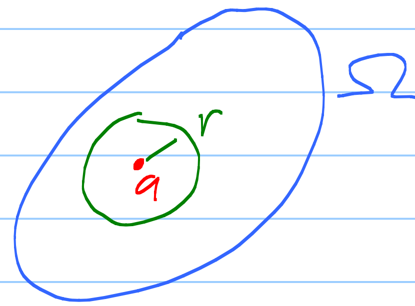
Pf of RMT: Let $\tilde{\mathcal{F}}$ = family of analytic fns $f: \Omega \to D_1(0)$ that are 1-1.

Lemma $\Rightarrow$ $\tilde{\mathcal{F}}$ not empty.

Pick $a \in \Omega$. Let $M = \sup\{|f'(a)| : f \in \tilde{\mathcal{F}}\}$

Step 1. Claim: $0 < M < \infty$

f 1-1 $\Rightarrow f'(a) \neq 0$.



Cauchy estimate: $|f'(a)| \leq \dfrac{1! \max_{C_r(a)} |f|}{r^1}$

$\leq \dfrac{1}{r}$. So $M \leq \dfrac{1}{r}$.

[Let r max out to see $|f'(a)| \leq \dfrac{1}{\text{dist}(a, b\Omega)}$.]

Step 2: Take seq $f_n \in \tilde{\mathcal{F}}$ such that $|f_n'(a)| \to M$ as $n \to \infty$. [$|f_n(z)| < 1$ $\forall n$, $z \in \Omega$.]

Montel's $\Rightarrow$ $\exists$ subseq $f_{n_k} \rightrightarrows F$ on Ω.

[F is the Riemann map!] analytic ✓

Step 3 Lemma $\Rightarrow$ $f_{n_k}'(a) \to F'(a)$

So $|F'(a)| = M$. ← not zero

$F'(a)$ not zero $\Longrightarrow$ F not constant.

Hurwicz $\Longrightarrow$ F is 1-1 on Ω.

Step 4 Claim: $F \in \tilde{\mathcal{F}}$. Note: $f_n : \Omega \to D_1(0)$

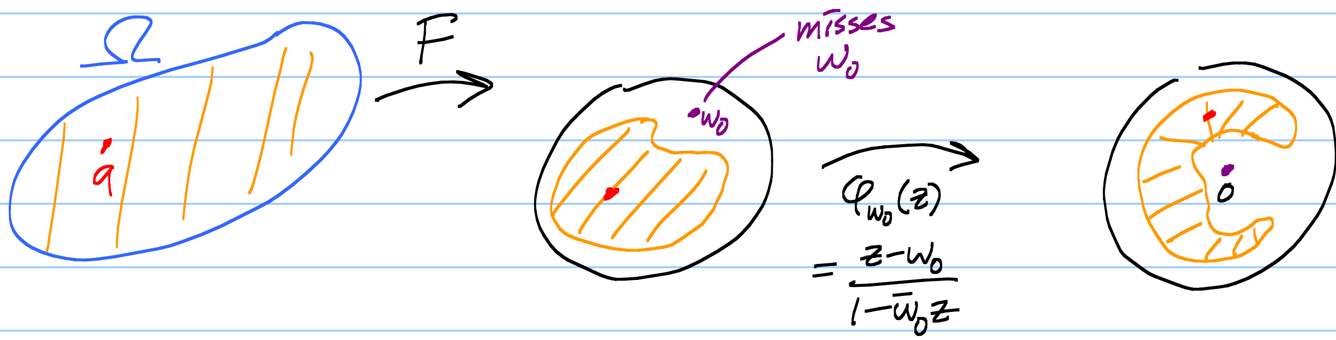
$|f_n(z)| < 1$

So $|F(z)| \le 1$.

Aha! If $|F(z_0)| = 1$ for some $z_0 \in \Omega$, Max Princ $\Longrightarrow$

$F \equiv$ const. $\lightning$ So $F : \Omega \to D_1(0)$. Get $F \in \tilde{\mathcal{F}}$ ✓

Step 5: Claim: F is onto $D_1(0)$. Suppose not.



Note: $\varphi_{w_0} \circ F$ is non-vanishing, 1-1, analytic.

So $\exists$ analytic g on Ω with $g^2 = \varphi_{w_0} \circ F$.

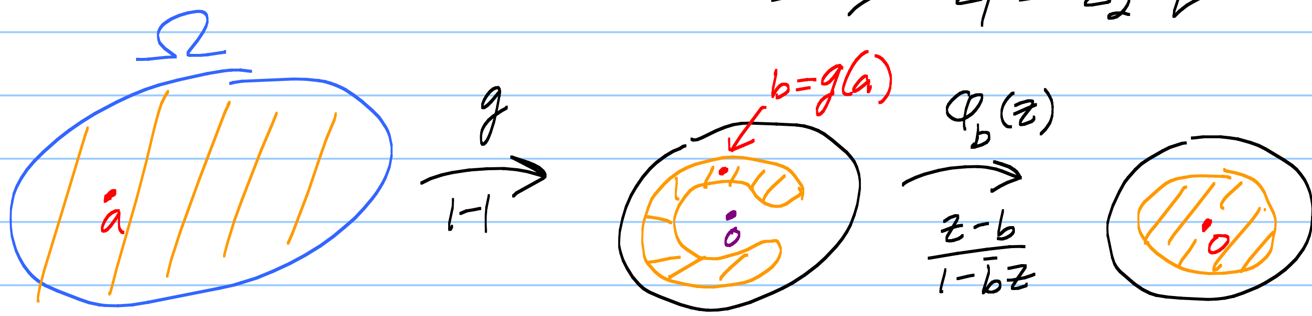
$[\,|g^2(z)| < 1 \Longrightarrow |g(z)| < 1.\,] \leftarrow g : \Omega \to D_1(0)$. ✓

Claim: $g \in \tilde{\mathcal{F}}$. Need g is 1-1.

Why: $g(z_1) = g(z_2) \Longrightarrow g(z_1)^2 = g(z_2)^2$

$$\varphi_{w_0}(F(z_1)) = \varphi_{w_0}(F(z_2)) \leftarrow \text{1-1}$$

$$\Longrightarrow z_1 = z_2 \checkmark$$



Claim: $\varphi_b \circ g \in \tilde{\mathcal{F}} \checkmark$ and derivative at a is strictly bigger in modulus than M. $\lightning$ So no such w_0.

Write $\tilde{F} = \varphi_b \circ g$.

Check that $F = G \circ \tilde{F}$

where $G = \varphi_{w_0}^{-1} \circ s \circ \varphi_b^{-1}$, $s(z) = z^2$

Note: $G : D_1(0) \to D_1(0)$

Cor to Schwarz: $|G'(0)| \le 1 \leftarrow$ if $=1$, then $G(z) = \underbrace{\lambda z}_{\text{1-1}}$, $|\lambda| = 1$.

Aha! Since $\underbrace{G \text{ is not 1-1}}_{s(z) \text{ is not 1-1}}$, we must have $|G'(0)| < 1$.

Finally,

$$F'(a) = G'(\underbrace{\tilde{F}(a)}_{0})\,\tilde{F}'(a)$$

$$|F'(a)| = |G'(0)|\,|\tilde{F}'(a)|$$

$\underbrace{\qquad}_{M} \qquad \underbrace{\qquad}_{<1} \qquad \underbrace{\qquad}_{\text{must be } > M!}$ ↯

No such w_0. F is <u>onto</u>. Done.

<u>Fact from proof</u>: $f : \Omega \to D_1(0)$, $\quad f(a) = b \neq 0$.

Then $\left| (\varphi_b \circ f)'(a) \right| > |f'(a)|$.

$$\left[\text{because } \quad |\varphi_b'(b)| = \frac{1}{1-|b|^2} > 1 \right]$$

<u>The</u> Riemann map associated to $a \in \Omega$: $f_a : \Omega \to D_1(0)$

$f(a) = 0$
$f'(a) = M$
1-1, onto
analytic

$f_a : \lambda F$

$f_a'(a) = \lambda F'(a) \qquad$ Take $\lambda = \dfrac{\overline{F'(a)}}{|F'(a)|}$

Makes <u>unique</u>.