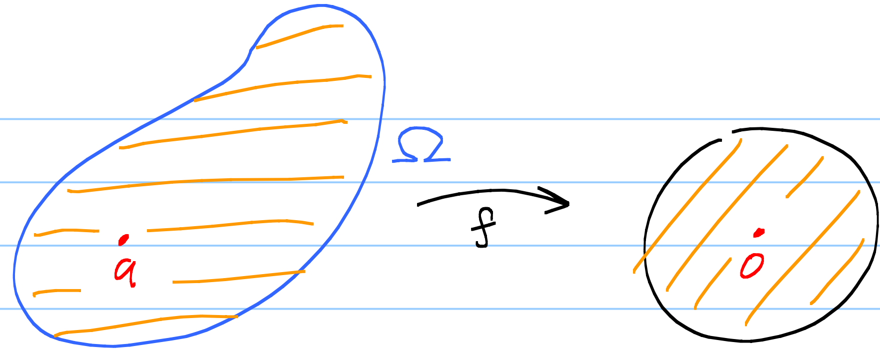
Lecture 32 Harmonic functions

HWK 7 due Tues in GS

Ave ≈ 91

Riemann map



The Riemann map $|f_a'(a)| = \text{Max}\{|h'(a)| : h : \Omega \xrightarrow{\text{analytic}} D_1(0)\}$

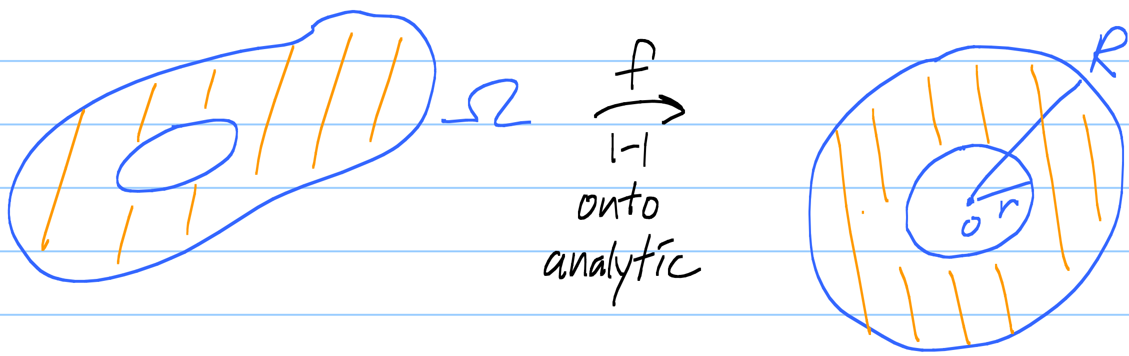
f_a solves this extremal problem

not assuming 1-1

Unique: $f_a'(a) \in \mathbb{R}^+$

Always: $f_a(a) = 0$.

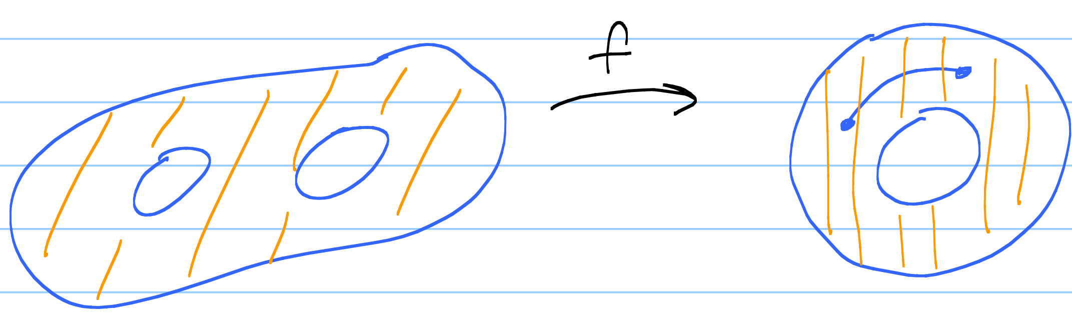
What about domains with holes?

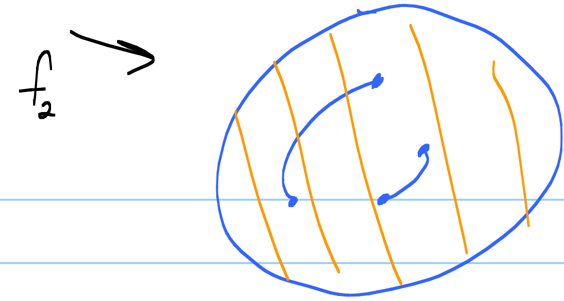


$\frac{r}{R}$ determines a "mapping class". $\frac{r}{R}$ = the "modulus" of Ω

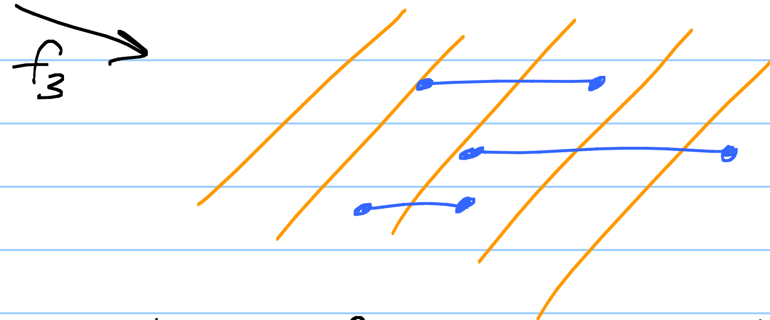
$\text{Aut}(\text{Annulus}) = \{\lambda z : |\lambda| = 1\} \cup \{\text{inversion} \sim \frac{1}{z}\}$

not transitive!





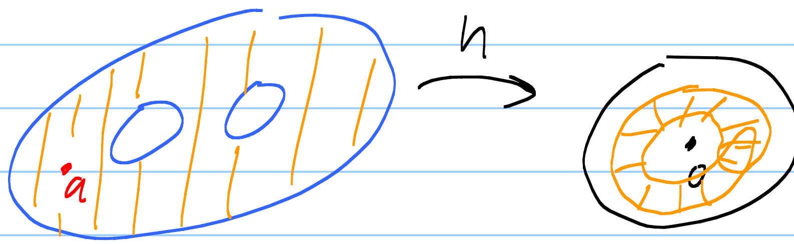
Disc minus circular slits



$\mathbb{C}$-horiz slits.

3-connected domains and above: finitely many automorphisms, perhaps (most likely) none.

Ahlfors map:



Extremal map f_a : $|f_a'(a)| = \mathrm{Max}\{|h'(a)| : \underset{\text{analytic}}{h: \Omega \to D_1(0)}\}$

Unique: $f_a'(a) \in \mathbb{R}^+$

f_a is <u>the</u> Ahlfors map. Like a Riemann map for n-connected domains. $(n>1)$.

Onto $D_1(0)$ ✓

Analytic ✓

Not 1-1. It is n-to-one, counting multiplicity.

Maps each boundary 1-1 onto unit circle.

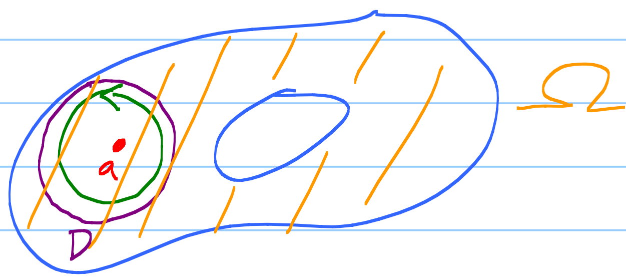
Harmonic fcn on a domain Ω is a C^2-smooth real valued fcn such that $\Delta u \equiv 0$.

Fact: $\iff$ u is locally the real part of an analytic fcn.

Remark: Can get global harmonic conjugates when Ω is simply connected.

[$\ln|z|$, no harm conj on $\mathbb{C}-\{0\}$.]

Important property: The averaging property



$$(*) \quad u(a) = \frac{1}{2\pi r}\int_0^{2\pi} u(a+re^{i\theta})\, \underbrace{r\,d\theta}_{ds}$$

Why: Get $f = u+iv$ on D, analytic.

Cauchy integral formula at a: $\underbrace{f(a)}_{u+iv} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{f(a+re^{i\theta})}_{u+iv}\,d\theta$

Take real part to get $(*)$.

Thm: Ave prop $\Rightarrow$ Max princicle.

Why: Suppose u has a local max at $z_0 \in \Omega$.

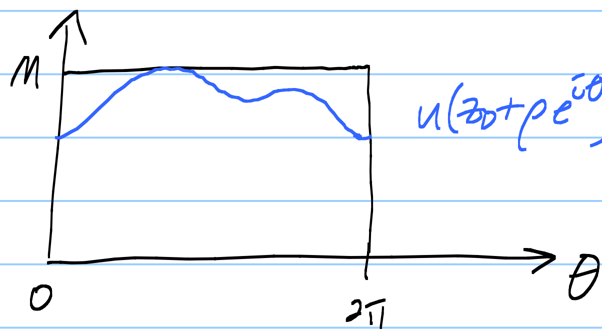
$\exists\ D_\varepsilon(z_0) \subset \Omega$ such that $u(z) \le u(z_0) = M$ on $D_\varepsilon(z_0)$.

Ave prop $\quad \underbrace{u(z_0)}_{M} = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta$

$0 < \rho < \varepsilon$

Thing inside $\int$ is $\leq M$ and continuous.

Calculus lemma. $\Longrightarrow u(z_0 + \rho e^{i\theta}) \equiv M, \quad \forall \theta, \; 0 < \rho < r.$



$u(z_0 + \rho e^{i\theta}) \leftarrow$ picture can't happen!

$u \equiv M.$

Conclude that $u \equiv M$ on $D_\varepsilon(z_0)$.

Aha! $\quad F = u_x - i u_y$ is analytic on Ω

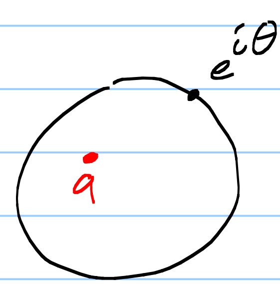
$u \equiv M$ on $D_\varepsilon \Longrightarrow F \equiv 0$ on $D_\varepsilon \Longrightarrow F \equiv 0$ on Ω,

$\Longrightarrow \nabla u \equiv 0$ on $\Omega \Longrightarrow u \equiv \underbrace{\text{const}}_{= M}$ on Ω. ✓

Want a "Cauchy integral formula" for harmonic fcns.

See "Something about Poisson and Dirichlet" Find a link below AFTERMATH on home page.

Poisson integral formula:



$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|a-e^{i\theta}|^2}\, u(e^{i\theta})\, d\theta$$

Simple Pf, easy to understand.

Formula true at $a=0$. Ave prop. ✓

Hmmm. $\varphi_{-a}(z) = \dfrac{z+a}{1+\bar{a}z}$: $D_1(0) \circlearrowleft$ $\overset{1-1}{\text{onto}}$ $\varphi_{-a}(0) = a$.

Fact: harmonic fcns composed with analytic fcns are harmonic.

Aha! Ave prop:

$$u(\underbrace{\varphi_{-a}(0)}_{a}) = \frac{1}{2\pi}\int_0^{2\pi} u(\underbrace{\varphi_{-a}(e^{i\theta})}_{e^{i\psi}})\, d\theta$$

$$e^{i\psi} = \frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}$$

Calculus: $$d\theta = \frac{1-|a|^2}{|a-e^{i\psi}|^2}\, d\psi$$

Get $$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})\, \frac{1-|a|^2}{|a-e^{i\psi}|^2}\, d\psi$$ ✓

Stein: p. 66: prob 11.