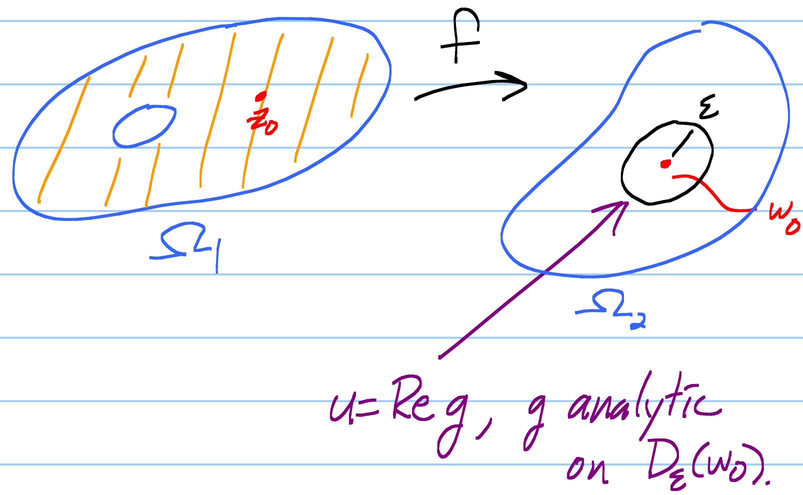
Lecture 33 The Dirichlet problem, Schwarz's theorem

Fact $f : \Omega_1 \to \Omega_2$ analytic, u harmonic on Ω_2. Then $u \circ f$ is harmonic on Ω_1.

Why:



$f(z_0) = w_0$

Ω_2 open: $\exists\, D_\varepsilon(w_0) \subset \Omega_2$
f analytic $\Rightarrow$ f cont.
So $\exists\, \delta > 0$ so that $D_\delta(z_0) \subset \Omega_1$ and $f(D_\delta(z_0)) \subset D_\varepsilon(w_0)$.

Finally $u \circ f = \underbrace{\text{Re}[g \circ f]}_{}$ on $D_\delta(z_0)$.

Real part of an analytic fcn.
So $u \circ f$ is harmonic on $D_\delta(z_0)$.

Poisson kernel technical points. To get Poisson formula

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |a|^2}{|a - e^{i\theta}|^2}\, u(e^{i\theta})\, d\theta \qquad (*)$$

for harmonic fcns, needed u harmonic on $D_R(0)$, $R > 1$. Get harmonic conj on $D_R(0)$, use ave prop., etc.

Easy extension: Suppose u is continuous on $\overline{D_1(0)}$ and harmonic on $D_1(0)$. Then u satisfies $(*)$ for $a \in D_1(0)$.

Why: $u_r(z) = u(rz)$, $0 < r < 1$, is harmonic on

$D_R(0)$ where $R = \frac{1}{r} > 1$. (*) holds for u_r.

Aha! Let $r \nearrow 1$. Use uniform continuity of u on $\overline{D_1(0)}$. Take limit. Get (*) for u.

Remark: Same trick works for analytic fcns continuous up to circle and Cauchy formula.

Hmmm. Famous problem: Given a continuous fcn $U(\theta)$ on $[0, 2\pi]$ with $U(0) = U(2\pi)$, is there a harmonic fcn u on $D_1(0)$ which extends continuously to $\overline{D_1(0)}$ such that $u(e^{i\theta}) = U(\theta)$?

Guess: Yes!

$$u(a) = \int_0^{2\pi} P(a, \theta)\, U(\theta)\, d\theta$$

where $P(a,\theta) = \frac{1}{2\pi} \frac{1 - |a|^2}{|a - e^{i\theta}|^2}$ is the Poisson kernel.

Dirichlet problem: Circular metal disc. Thermal equillibrium. Know temp on boundary. Want to know temp inside.

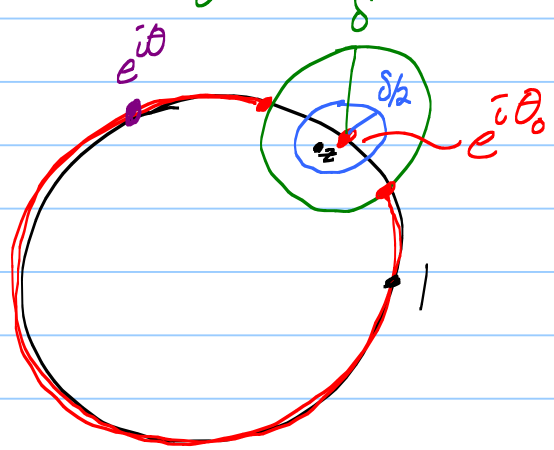
Schwarz's thm: Poisson formula solves D. prob.

3 Key properties of $P(a,\theta)$

1) $P(a,\theta) > 0$ ✓ formula

2) $\int_0^{2\pi} P(a,\theta)\, d\theta = 1$ ✓ (*) using $u(z) \equiv 1$.

3)



$I_\delta = \{\theta : e^{i\theta} \notin D_\delta(e^{i\theta_0})\}$

$$P(z,\theta) \le \frac{4}{\pi\delta^2}(1-|z|)$$

$\longrightarrow 0$ uniformly on I_δ

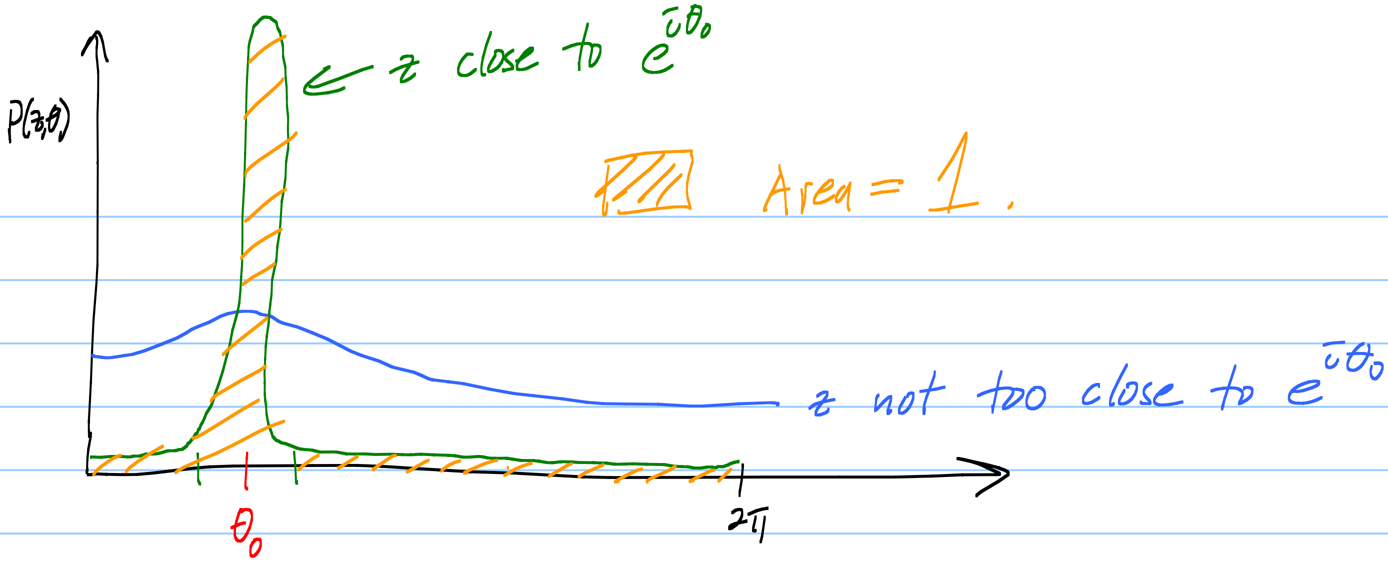
when $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$.

(3) easy: $$\frac{1}{2\pi}\frac{1-|z|^2}{|z-e^{i\theta}|^2} \le \frac{1}{2\pi}\frac{1-|z|^2}{\left(\frac{\delta}{2}\right)^2} = \frac{1}{2\pi}\frac{(1-|z|)\underbrace{(1+|z|)}_{\le 2}}{\left(\frac{\delta}{2}\right)^2}$$

$$< \frac{4}{\pi\delta^2}(1-|z|)$$ ✓

4) $P(z,\theta)$ is harmonic in $z \in D_1(0)$.

because $$P(z,\theta) = \frac{1}{2\pi}\,\mathrm{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]$$



Pf: Define $u(z) = \int_0^{2\pi} P(z,\theta)\, U(\theta)\, d\theta$

$$= \mathrm{Re}\, \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, U(\theta)\, d\theta}_{\text{analytic: HWK 2: p. 1.}}$$

u is harmonic on $D_1(0)$. ✓

Step 2: Show $\lim_{z\to e^{i\theta_0}} u(z) = U(\theta_0)$.

$$u(z) - U(\theta_0) = \int_0^{2\pi} P(z,\theta)\left[U(\theta) - U(\theta_0)\right] d\theta = I$$

because $\int_0^{2\pi} P(z,\theta)\, d\theta = 1$.

Let $\varepsilon > 0$. $\exists\ \delta > 0$ such that $|U(\theta) - U(\theta_0)| < \frac{\varepsilon}{2}$

when $e^{i\theta} \in D_\delta(e^{i\theta_0})$.

Now $I = \int_{I_\delta} \underbrace{P(z,\theta)}_{\text{small}} \left(U(\theta) - U(\theta_0)\right) d\theta \qquad I_1$

$$+ \int_{[0,2\pi]-I_\delta} P(z,\theta)\underbrace{\big(U(\theta) - U(\theta_0)\big)}_{\text{small}}\,d\theta \qquad I_2$$

Suppose $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$.

$$|I_1| \le \frac{4}{\delta^2\pi}(1-|z|)\; 2M \underbrace{\text{Length}(I_\delta)}_{<2\pi} \longrightarrow 0 \text{ as } z \to e^{i\theta_0}$$

inside set.

$M = \max_{[0,2\pi]} |U(\theta)|$

$$|I_2| \le \int_{[0,2\pi]-I_\delta} P(z,\theta) \underbrace{|U(\theta) - U(\theta_0)|}_{<\frac{\varepsilon}{2}}\,d\theta$$

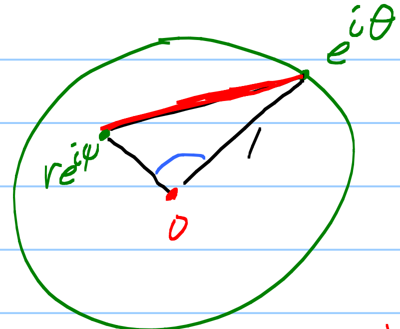
P possitive!

$$< \frac{\varepsilon}{2}\int_{[0,2\pi]-I_\delta} P(z,\theta)\,d\theta < \frac{\varepsilon}{2}\underbrace{\int_0^{2\pi} P(z,\theta)\,d\theta}_{=1\,!}$$

$P > 0$.

$< \frac{\varepsilon}{2}$. Done!

Other formulas



blue angle $= \psi - \theta$

red line $=$

$$\sqrt{1 + r^2 - 2r\cos(\psi-\theta)}$$

$$u(re^{i\psi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\psi-\theta)} \varphi(\theta)\, d\theta$$

$D_R(z_0)$ version:

$$u(z_0 + re^{i\psi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2-r^2)}{R^2-2rR\cos(\psi-\theta)+r^2} u(z_0+Re^{i\theta})\, d\theta$$

Note: $r=0$ gives ave property.

Explosion of theorems!