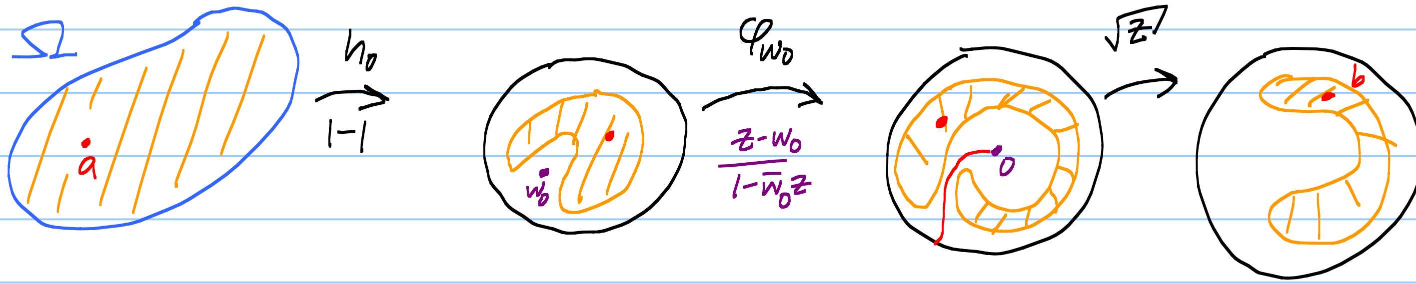
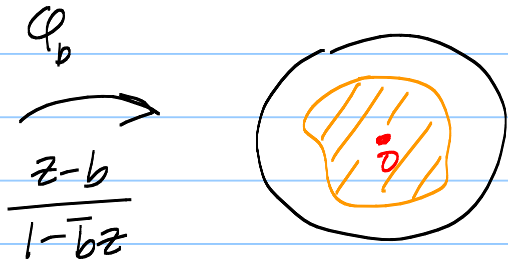
Lecture 34 Applications of the Poisson formula

Friday odds and ends: Koebe's contructive pf of the RMT.





Construction: Composition

$h_1 : \ |h_1'(a)| > |h_0(a)|$

Repeat! Get seq h_j. Koebe: $h_j \rightrightarrows f_a$ ← Riemann map.

Much better ways to compute f_a are known now.

Remarks about maximum principles

Last time: Ave prop for harm fcns $\Rightarrow$ Max princ for harm fcns.

Note: Max princ for harm fcns $\Rightarrow$ Min princ for harm fcns.

Pf: Max princ for $-u \Rightarrow$ min princ for $+u$.

Remark: Max modulus thm $\Rightarrow$ Max princ for harm fcns.

Pf: Suppose harm fcn u has local max at z_0.

Shrink a disc $\overline{D_r(z_0)} \subset \Omega$ so that

$u(z) \leq u(z_0) = M$ on $\overline{D_r(z_0)}$.

Get $f = u + iv$ analytic on $D_r(z_0)$.

<u>Trick</u>: $e^f = e^{u+iv} = e^u e^{iv}$

So $\boxed{|e^f| = e^u}$ ← has max at z_0!

So MMT $\Rightarrow$ $e^f \equiv$ const. on $D_r(z_0)$

Red box: $u = \operatorname{Ln}|e^f| \equiv$ const too.

Know $u \equiv c$ on $D_r(z_0) \Rightarrow u \equiv c$ on domain Ω.

<u>Remark</u>: No Min princ for analytic fcns. e.g. z on $D_1(0)$.

However, there is a Min modulus thm for nonvanishing analytic fcns: Max princ for $1/f \Rightarrow$ Min princ for f!

<u>Big moment</u>: Ave prop $\Longrightarrow$ harmonic.

<u>Thm</u>: Suppose u is a continuous real valued fcn on a domain Ω. If u satisfies the ave prop: $\overline{D_r(z_0)} \subset \Omega$: $u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$, then u is harmonic.

<u>Pf</u>: Being harmonic is a local property, so can restrict attention to a disc. $\frac{1}{R}(z - z_0)$ maps $D_R(z_0) \longleftrightarrow D_1(0)$. Can assume u satisfies ave prop on $D_r(0)$, $r > 1$.

Classic trick: Define

$$\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta$$

Know $\tilde{u}$ harmonic on $D_1(0)$, continuous on $\overline{D_1(0)}$,

and $\tilde{u}(e^{i\theta}) \equiv u(e^{i\theta})$.

Know: harm fcns satisfy ave prop.

Aha! $u - \tilde{u}$ satisfies ave prop!

Ave prop $\Rightarrow$ max princ $\Rightarrow$ $u - \tilde{u}$ assumes max value on $C_1(0)$, where it is zero.

So $u - \tilde{u} \leq 0$ on $\overline{D_1(0)}$.

Repeat using $\tilde{u} - u$ in place of $u - \tilde{u}$. Get $\tilde{u} - u \leq 0$.

So $u \equiv \tilde{u} \leftarrow$ harmonic. ✓

Riemann removable singularity thm for harmonic fcns.

Pf: Can assume u harmonic on $D_R(0) - \{0\}$, $R > 1$.

We assume u is bounded there. Show 0 is removable.

Let $\tilde{u}$ = Poisson integral of bndry values of u on $C_1(0)$.

Know $\tilde{u} \equiv u$ on $C_1(0)$, $\tilde{u}$ harm on $D_1(0)$, cont on $\overline{D_1(0)}$.

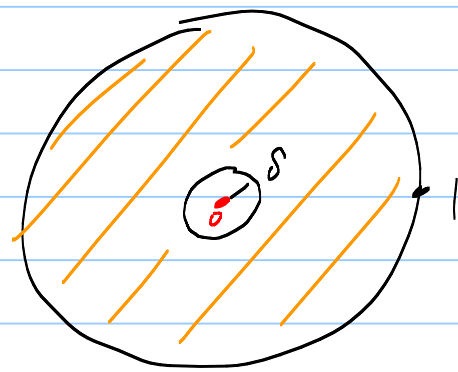
Sneaky trick: Define $v = u - \tilde{u} + \varepsilon \underbrace{\operatorname{Ln}|z|}$ ← $\varepsilon > 0$

harmonic on $\mathbb{C} - \{0\}$,
$\equiv 0$ on $C_1(0)$,
$\to -\infty$ as $|z| \to 0$.

Have: v harmonic on $D_1(0) - \{0\}$,
v is continuous up to $C_1(0)$ and $v \equiv 0$ on $C_1(0)$.

Key: u bounded on $D_1(0) - \{0\} \implies v(z) \to -\infty$ as $z \to 0$.

Want to use max princ.



Choose $0 < \delta < 1$ such that $v(z) < -1$ when $|z| \leq \delta$, $z \neq 0$.

Apply max princ to v on $\overline{D_1(0)} - D_\delta(0)$.

Max occurs on bndry: $v \leq \operatorname{Max}\{0, -1\}$ on domain.

Get $v \leq 0$ on $\overline{D_1(0)} - D_\delta(0)$.

Can let $\delta \searrow 0$. Get $v \leq 0$ on $\overline{D_1(0)} - \{0\}$.

$v = u - \tilde{u} + \varepsilon \operatorname{Ln}|z| \leq 0$

Aha! Can let $\varepsilon \searrow 0$!

Get $u - \tilde{u} \leq 0$.

Repeat using $\tilde{u}-u$ in place of $u-\tilde{u}$.

So

defining $u(0)=\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta$

makes u harm on $D_1(0)$.

Fun fact: Solution to the Dirichlet prob on $D_1(0)$ for polynomial data $p(x,y)$ on $C_1(0)$ is a polynomial harmonic fcn.

Pf: $p(x,y) = p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = P(z,\bar{z})$

$$= \sum_{n,m=0}^{N} c_{nm} z^n \bar{z}^m$$

Trick:

$$z^n \bar{z}^m = \begin{cases} 1 & \text{on } C_1 \quad n=m \\ \underbrace{z^n \bar{z}^n}_{1} \bar{z}^{m-n} = \bar{z}^{m-n} & \text{on } C_1 \quad m>n \\ z^{n-m} \underbrace{z^m \bar{z}^m}_{1} = z^{n-m} & \text{on } C_1 \quad m<n \end{cases}$$