Lecture 36 Weak averaging property and Schwarz reflection for harmonic fcns

HWK 8 due Thurs, 4/21 in GS

$$P(z,\theta) = \frac{1}{2\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] \qquad z(\theta) = e^{i\theta} \qquad z'(\theta) = ie^{i\theta}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, u(e^{i\theta}) \frac{\overbrace{z'(\theta)\, d\theta}^{dz}}{z'(\theta) = ie^{i\theta}}$$

continuous on C

$$= \frac{1}{2\pi i}\int_C \frac{u(w)}{w-z}\, dw + \frac{z}{2\pi i}\int_C \frac{u(w)/w}{w-z}\, dw$$

HWK 2 : Prob 1 $\Rightarrow$ analytic in z.

<u>Note</u>: By differentiating under Poisson integral, get "Cauchy" estimates for harmonic fcns. Use them to prove:

<u>Thm</u>: Suppose u_n harmonic on a domain Ω and $u_n \rightrightarrows u$. Then u is harmonic and

$$\frac{\partial^{k+m} u_n}{\partial x^k \partial y^m} \rightrightarrows \frac{\partial^{k+m} u}{\partial x^k \partial y^m}$$

<u>Theorem</u>: Suppose u is

1) Continuous on Ω

2) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ exist on Ω, and

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$ on Ω.

Then u is harmonic $\Longleftrightarrow$ $\underbrace{\text{locally} = \operatorname{Re} f,\ f \text{ analytic}}_{C^\infty \text{ smooth!}}$

Pf: Thm local, So we can assume hyp valid on $D_R(0)$, $R > 1$.

Let $\tilde{u}$ = Poisson integral of $u(e^{i\theta})$.

Want to see that $u \equiv \tilde{u}$. If not, then

$\exists\ z_0 \in D_1(0)$ $\quad u(z_0) - \tilde{u}(z_0) \neq 0$. Can assume

positive by replacing $u - \tilde{u}$ by $\tilde{u} - u$.

Say $u(z_0) - \tilde{u}(z_0) = m > 0$.

Note $u - \tilde{u} \equiv 0$ on $C_1(0)$.

Sneaky trick: $\varepsilon|z|^2 = \varepsilon(x^2 + y^2)$ is small on $\overline{D_1(0)}$

and $\Delta(\varepsilon|z|^2) = 4\varepsilon > 0$ there.

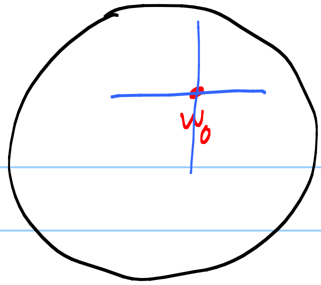
Let $v = u - \tilde{u} + \varepsilon|z|^2$ $\Leftarrow$ want a positive max in $D_1(0)$.

Aha! Take $0 < \varepsilon < \frac{m}{2}$.

Then $v(z_0) = m + \varepsilon|z_0|^2 \geq m$

$v(e^{i\theta}) = 0 + \varepsilon < \frac{m}{2}$

max v on $\overline{D_1(0)}$ occurs at some pt $w_0 \in D_1(0)$.



2nd derivative test in horiz and vert directions at w_0

[$\tilde{u}''(t_0) > 0$, strict local min]

Aha! Since $v(w_0)$ is a max,

$$\frac{\partial^2 v}{\partial x^2}(w_0) \le 0$$

$$+ \quad \frac{\partial^2 v}{\partial y^2}(w_0) \le 0$$

$$\Delta v\,(w_0) \le 0 \quad ↯ \quad \Delta v = 4\varepsilon > 0.$$

No such z_0. $u - \tilde{u} \equiv 0$. Done!

Elliptic regularity: Δu smooth $\Rightarrow$ u smooth.

Weak averaging property A continuous fcn u on a domain Ω satisfies the W.A.P. if, given any point $z_0 \in \Omega$, there is $r > 0$ such tha $D_r(z_0) \subset \Omega$ and $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\,d\theta$ for $0 < \rho < r$.

Remark: r could be small. $\left(r < \operatorname{dist}(z_0, b\Omega) \right)$

Thm: W.A.P. $\Rightarrow$ u harmonic.

Pf: Local thm. Can assume u continuous on $D_R(0)$, $R>1$ and satisfies WAP.

Let $\tilde{u}$ = Poisson integral of $u(e^{i\theta})$.

Let $U = \tilde{u} - u$.

Note $U \equiv 0$ on $C_1(0)$.

U satisfies WAP too [$\tilde{u}$ is harmonic!]

Want $U \equiv 0$ on $D_1(0)$. Assume $U(z_0) \neq 0$.

Can assume $U(z_0) > 0$ by replacing $\tilde{u}-u$ with $u-\tilde{u}$.

Then U will a max M on $\overline{D_1(0)}$, $M>0$.

Since $U \equiv 0$ on $C_1(0)$, max occurs inside $D_1(0)$.

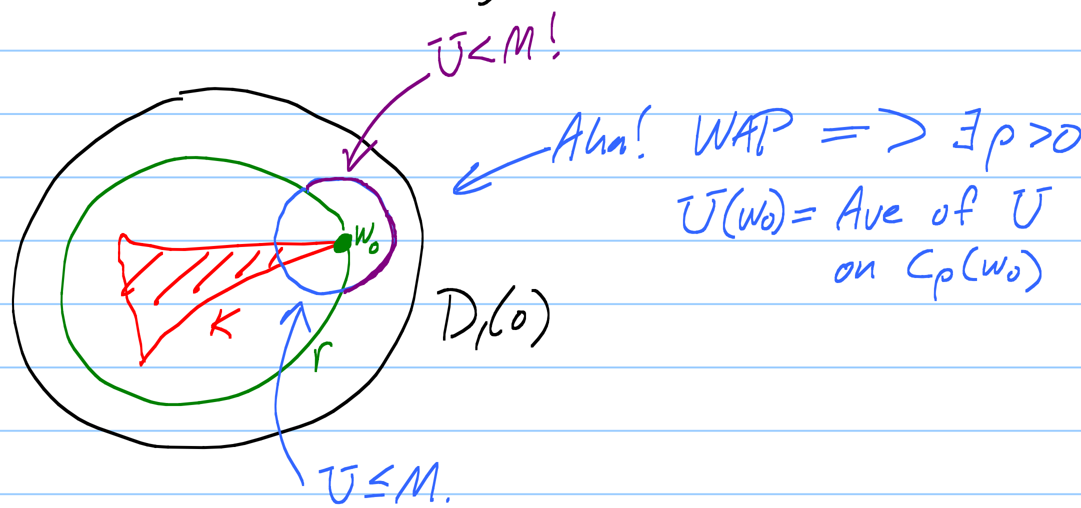
Hmmm. Let $K = \{z \in \overline{D_1(0)} : U(z) = M\}$.

K is closed (by continuity) and bounded

<u>inside</u> $D_1(0)$. K is compact.

$|z|$ is a continuous fcn on K, so it has a max r

on K.

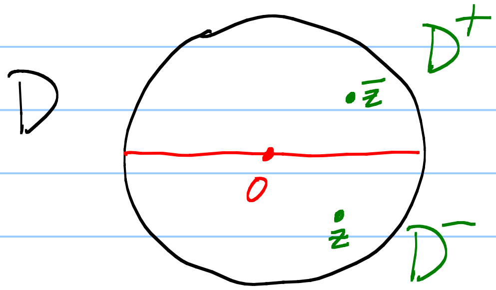


See Ave $\tilde{U}$ of $C_\rho(w_0) < M$. ↯

So no such z_0 where $\tilde{u}-u$ (or $u-\tilde{u}$) not zero.

Conclude $u \equiv \tilde{u}$ on $D_1(0)$. ✓

Schwarz reflection principle for harmonic fcns.



Suppose u is a real valued harmonic fcn on D^+ which is continuous up $D \cap \mathbb{R}$. If $u \equiv 0$ on $D \cap \mathbb{R}$,

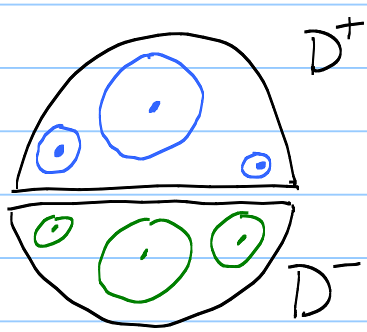
then

$$\tilde{U}(z) = \begin{cases} u(z) & z \in D^+ \\ 0 & z \in D \cap \mathbb{R} \\ -u(\bar{z}) & z \in D^- \end{cases}$$

is harmonic on D.

Note: $-u(\bar{z})$ is harmonic on D^-.

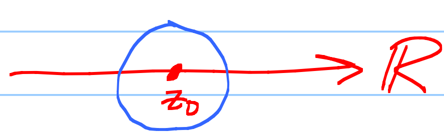
Pf:



WAP on D^+ ✓

WAP on D^- ✓

Last step



$$0 = u(z_0) = \underbrace{\int_{Top} + \int_{Bot}}_{\text{Cancel!}}$$

WAP holds at $z_0 \in \mathbb{R}$ too. ✓