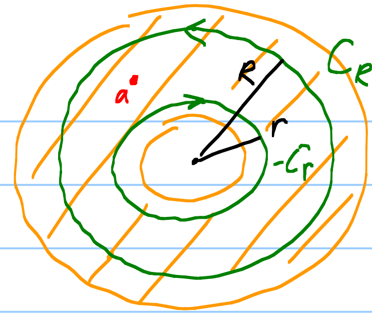
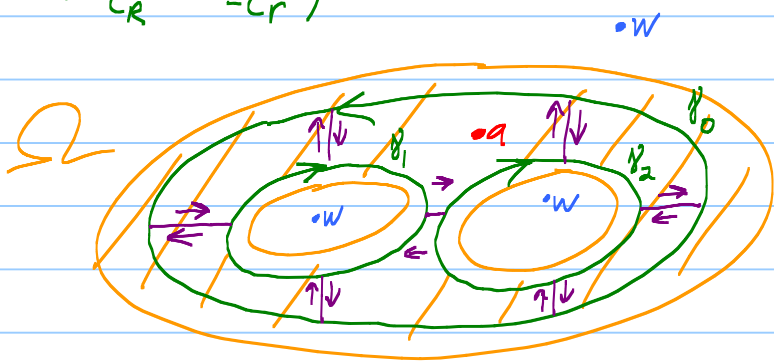
# Lecture 37 The general Cauchy theorem

Cauchy theorem for an annulus




Cauchy int formula too

$$\left(\int_{C_R} + \int_{-C_r}\right) f\, dz = 0$$




$$\Gamma = \underbrace{\gamma_0 \cup \gamma_1 \cup \gamma_3}_{\gamma\text{'s closed curves in } \Omega} \leftarrow \text{cycle } \Gamma$$

$$\int_\Gamma f\, dz = \sum \int_{\gamma_j} f\, dz$$

$f$ analytic on $\Omega$, then $\int_\Gamma f\, dz = 0$. And

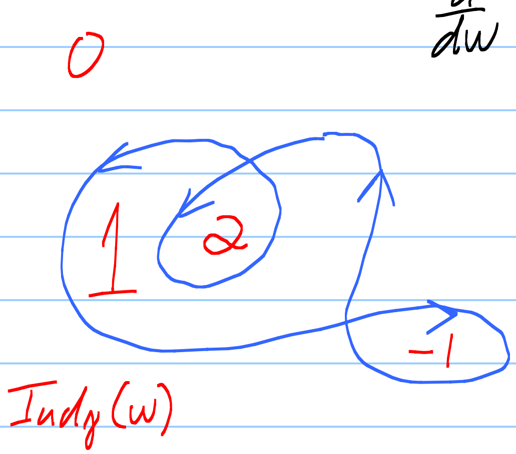
$$f(a)\, \mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\, dz$$

where $\mathrm{Ind}_\Gamma(a) = \sum \underbrace{\mathrm{Ind}_{\gamma_j}(a)}_{\frac{1}{2\pi i}\int_{\gamma_j}\frac{dz}{z-a}}$

$$\frac{1}{2\pi i}\int_{\gamma_j}\frac{dz}{z-a} = \text{"winding number of } \gamma_j \text{ about } a\text{"}$$

<u>Recall</u>: Closed curve $\gamma$ is such that $\mathbb{C} - tr(\gamma)$ is a union of open connected components. $\mathrm{Ind}_\gamma(w) \equiv$ const on each component because

$$\frac{d}{dw}\mathrm{Ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-w)^2}$$




$$= \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}\left[\frac{-1}{z-w}\right] dz = 0$$

Important assumption to get a general Cauchy thm:

$$\mathrm{Ind}_\Gamma(w) = 0 \quad \text{for } \underline{\text{all}} \quad w \in \mathbb{C} - \Omega.$$

<u>Fancy word</u> $\Gamma$ is "homologous to zero" in $\Omega$.

Write $\Gamma \sim 0$ or $\Gamma \sim_\Omega 0$.

<u>General Cauchy Thm</u>: $\Gamma \sim 0$ and $f$ analytic on $\Omega$.

1) $\int_\Gamma f \, dz = 0$

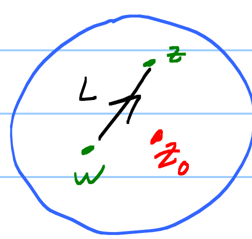
2) $f(a)\,\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a}\, dz \qquad a \in \Omega - \mathrm{tr}(\Gamma)$

<u>Pf</u>: (Ahlfors)

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$

<u>Step 1</u> $g$ is continuous on $\Omega \times \Omega$.

Clear at $(z_0, w_0)$, $z_0 \neq w_0$. Only to check near pts $(z_0, z_0)$.




$$\left| g(z,w) - g(z_0,z_0) \right| = \Bigg| \underbrace{\frac{f(z)-f(w)}{z-w}}_{\frac{1}{z-w}\int_L f'(\zeta)\,d\zeta} - \underbrace{f'(z_0)}_{\frac{1}{z-w}\int_L f'(z_0)\,d\zeta} \Bigg|$$

$$= \left| \frac{1}{z-w} \int_L \left[ f'(\zeta) - f'(z_0) \right] d\zeta \right|$$

$$\leq \frac{1}{|z-w|} \underbrace{\left( \max_{\zeta \in L} |f'(\zeta) - f'(z_0)| \right)}_{\substack{f' \text{ continuous.} \\ \text{Can easily make small} \checkmark}} \underbrace{|z-w|}_{\text{length}(L)}$$

Step 2 Define $h(z) = \int_\Gamma g(z,w)\,dw$

Plan: Extend $h$ to $\mathbb{C}$ as an entire fcn that tends to zero at $\infty$. Liouville's $\Rightarrow$ Cauchy thm!

Step 3 $h$ is analytic on $\Omega$.

a) clear that $h$ is continuous on $\Omega$.

[why: $g$ cont on $\Omega \times \Omega$. $tr(\Gamma) = \bigcup tr(\gamma_j)$ is compact. $g$ cont on compact $\overline{D_\delta(z_0)} \times tr(\Gamma)$ is unif continuous. Get $h$ is cont.]

b) $\Delta \subset \Omega$ triangle.

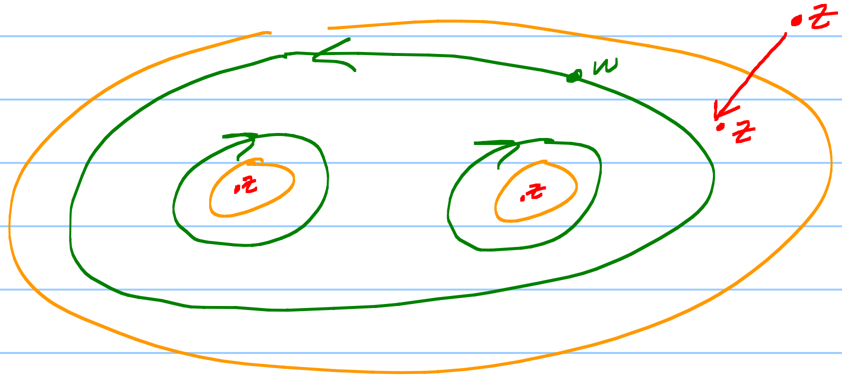
$$\int_\Delta h\,dz = \int_\Delta \left( \int_\Gamma g(z,w)\,dw \right) dz$$

$$\underset{\text{Fubini}}{=} \int_\Gamma \left( \underbrace{\int_\Delta g(z,w)\,dz}_{=0} \right) dw = 0$$

Aha! When $w$ fixed, $g(z,w)$ has a removable sing of $z$ at $z=w$.

Morera $\Rightarrow$ $h$ analytic!

<u>Step 4</u> Extend $h$ to $\mathbb{C}$




$\mathrm{Ind}_\Gamma(z) = 0$

$$h(z) = \int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw = f(z)\underbrace{\int_\Gamma \frac{1}{z-w}\,dw}_{=0} - \int_\Gamma \frac{f(w)}{z-w}\,dw$$

"outside" $\Gamma$

$$\tilde{h}(z) = -\int_\Gamma \frac{f(w)}{z-w}\,dw \quad \text{on} \quad \tilde{\Omega} = \{z \in \mathbb{C} - tr(\Gamma) \text{ where } \mathrm{Ind}_\Gamma(z) = 0\}$$

<u>Claim:</u> $\tilde{h}$ defines an extension of $h$ to $\mathbb{C}$ as an entire fcn.

<u>Facts</u>: 1) $\tilde{\Omega}$ is open.

2) $\Omega \subset \mathbb{C} - \tilde{\Omega}$

3) $\tilde{h}$ is analytic on $\tilde{\Omega}$ (HWK 2; prob 1).

4) $h = \tilde{h}$ on overlap $\Omega \cap \tilde{\Omega}$

So $\tilde{h}$ extends $h$ to $\mathbb{C}$, analytic. Call ext $h$.

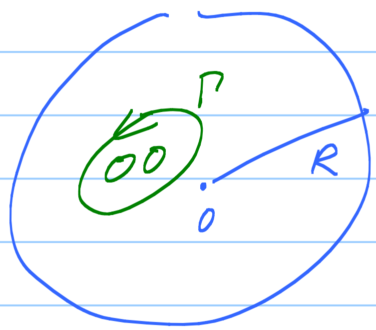
<u>Claim</u> $h(z) \to 0$ as $z \to \infty$.

<u>Why</u>: $tr(\Gamma) \subset D_R(0)$ for some $R > 0$.

So $Ind_\Gamma(w) = 0$ for $|w| > R$.

So, for $|z| > R$:

$$h(z) = \tilde{h}(z) = -\int_\Gamma \frac{f(w)}{z-w}\, dw$$




$$|h(z)| \le \max_{w \in \Gamma} \frac{|f|}{|z-w|} \text{ length}(\Gamma)$$

$$\le \frac{M}{|z|-R} \text{ length}(\Gamma)$$

$$\to 0 \text{ as } |z| \to \infty.$$

Liouville's $\Rightarrow h \equiv 0$.

So $$\int_\Gamma \frac{f(z) - f(w)}{z-w}\, dw = 0 \qquad z \in \Omega - tr(\Gamma)$$

$$f(z) \underbrace{\int_\Gamma \frac{1}{z-w}\, dw}_{2\pi i\, Ind_\Gamma(z)} = \int_\Gamma \frac{f(w)}{z-w}\, dw \quad \checkmark$$

Cauchy integral formula ✓ CIF $\Rightarrow$ Cauchy Thm