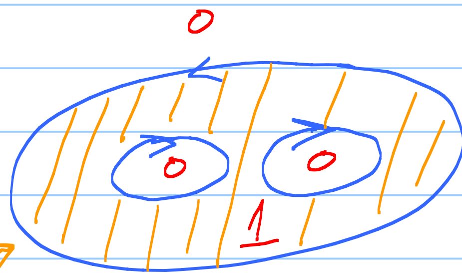
Lecture 39 Mittag-Leffler and Weierstraß theorems

HWK 8 due Thurs (last one)

Simple cycle:



"inside Γ"

$\mathrm{Ind}_\Gamma = 0$ or 1, need to add

$\boxed{\{z : \mathrm{Ind}_\Gamma(z) = 1\} \text{ is connected.}}$

Improved residue theorem Suppose Ω is a simply connected domain and A is a discrete subset of Ω and f is analytic on $\Omega - A$. If γ is a closed curve in $\Omega - A$, then

$$\int_\gamma f\,dz = 2\pi i \sum_{a\in A} \mathrm{Ind}_\gamma(a)\,\mathrm{Res}_f(a)$$

finitely many $\neq 0$, so finite sum

allow ess. sing.!

Remark Improved residue thms $\Longrightarrow$ improved Arg princ, improved Rouché's, ...

e.g. Suppose f, g analytic inside and on a simple cycle Γ except at finitely many points inside where it has poles. No zeroes on Γ, too.

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}\,dz = \begin{pmatrix}\#\text{ zeroes}\\ \text{inside } \Gamma\end{pmatrix} - \begin{pmatrix}\#\text{ poles}\\ \text{inside } \Gamma\end{pmatrix}$$

with mult.

$=$ # times $f(z)$ goes around origin in the counterclockwise <u>as</u> z goes around curves γ_j in Γ (in the direction of each γ_j).

Rouché's: $|f(z)-g(z)| < |f(z)|$ on $tr(\Gamma)$, then

f and g have the same number of zeroes inside Γ (with mult) $\leftarrow$ no poles

(#zeroes) $-$ (#poles) same for f, g $\leftarrow$ allow poles.

<u>Mittag-Leffler thm</u> Given a seq $\{a_n\}_{n=1}^{\infty}$ of distinct pts in $\mathbb{C}$ such that $a_n \to \infty$ as $n \to \infty$ (A discrete), and principal parts $R_n(z) = \dfrac{A_{N_n,n}}{(z-a_n)^{N_n}} + \cdots + \dfrac{A_{1,n}}{(z-a_n)}$,

there exists a meromorphic fcn f on $\mathbb{C}$ with poles exactly at the a_n with princ part R_n there.

[$f - R_n$ has a removable sing at a_n.]

<u>Def^n</u>: f is called meromorphic on a domain Ω if there is a discrete set of pts $A \subset \Omega$, f is analytic on $\Omega - A$, f only has poles at pts in A.
Note: A could be empty.

Think: Give f the value ∞ at pts in A.

Get $f : \Omega \to \hat{\mathbb{C}}$ "analytic" from the point of view of $\hat{\mathbb{C}}$ as a complex manifold (Riemann surface).

[Coordinate chart at ∞:

$$\underbrace{\{z : |z| > R\} \cup \{\infty\}}_{\text{polar cap}} \xrightarrow{1/z} D_{1/R}(0)$$]

Pf of M-L: First try: $f = \sum_{n=1}^{\infty} R_n(z)$

Ouch! Might not converge.

Second try: $f = \sum_{n=1}^{\infty} \left[R_n(z) - \underbrace{p_n(z)}_{} \right]$

Taylor poly for $R_n(z)$ such that $|R_n(z) - p_n(z)| < \frac{1}{2^n}$ on $\overline{D_{r_n}(0)}$ where $r_n = \frac{|a_n|}{2}$

Claim: f solves M-L problem.

Pf: Take big disc $D_R(0)$. $\exists\, N$ such that $r_n = \frac{|a_n|}{2} > R$ if $n > N$.

$$f(z) = \underbrace{\sum_{n=1}^{N} \left(R_n(z) - p_n(z) \right)}_{\substack{\text{rational fcn with} \\ \text{correct princ parts} \\ \text{in } D_R(0)}} + \sum_{n=N+1}^{\infty} \left(\underbrace{R_n(z) - p_n(z)}_{eeeeeee} \right)$$

$|eeee| < \frac{1}{2^n}$ on $D_R(0)$

Weierstraß M-test $\Rightarrow$ this Σ conv unif to analytic fcn on $D_R(0)$.

Remark M-L Thm is true on any domain Ω, discrete $A \subset \Omega$.

Lars Hörmander: Several complex variables: Chap 1.

Weierstraß thm Given seq $\{a_n\}_{n=1}^{\infty}$ as in M-L thm, and positive integers m_n, there is an entire fcn f that has zeroes of mult m_n at a_n's and no other zeroes.

Classic proof $f(z) = \prod_{n=1}^{\infty} \underbrace{\left(1 - \frac{z}{a_n}\right)^{m_n} e^{P_n(z)}}_{\substack{\text{need terms to} \rightarrow 1 \\ \text{fast enough}}}$ $\leftarrow$ Weier polys

My proof M-L $\Rightarrow$ Weier via general residue thm.

Hmmm. $\frac{f'}{f} = \frac{m_n}{z - a_n} + (\text{analytic})$ near a_n

Hmmm. $f(z) = e^{\log f} = e^{\int_\gamma \frac{f'}{f} dw}$

Aha! M-L says $\exists$ mero F with poles at a_n with p.p. $\frac{m_n}{z-a_n}$.

"Define" $$f(z) = \exp\left(\int_{\gamma_a^z} F(w)\,dw\right)$$

where γ_a^z is a curve in $\mathbb{C} - \{a_n\}_1^\infty$ from a fixed pt a to z.

Step 1: f is well defined.

$$\frac{f(z)}{\tilde{f}(z)} = \frac{\exp\left(\int_{\gamma_a^z} F\,dw\right)}{\exp\left(\int_{\tilde{\gamma}_a^z} F\,dw\right)} = \exp\left[\underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right)}_{\int_\gamma} F\,dw\right]$$

$\gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ $\uparrow$ closed

$$= \exp\left[2\pi i \underbrace{\sum_1^\infty \mathrm{Ind}_\gamma(a_n)\cdot m_n}_{\substack{\text{finite sum} \\ = \text{integer}}}\right] = 1$$

Next time: show f does it.