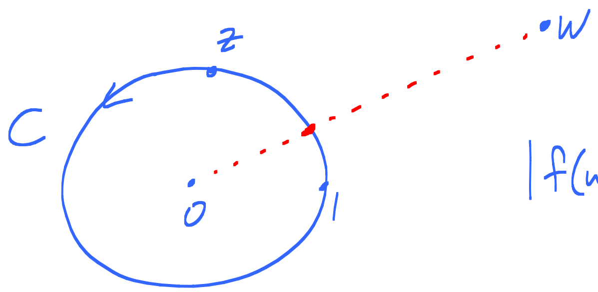
Review lecture 1

Final exam a take-home exam. Available noon on Tuesday. Due 11:59 pm Wed. Open book, notes only. 3 hours max.

1. Suppose that ϕ is a continuous function on the unit circle in the complex plane, and let C denote the unit circle parametrized via $z(t) = e^{it}$, $0 \le t \le 2\pi$. Let $\Omega = \{w \in \mathbb{C} : |w| > 1\}$. For $w \in \Omega$, define

$$f(w) = \int_C \frac{\phi(z)}{z - w}\, dz.$$

What kind of singularity does f have at infinity? Use careful estimates and explain.



$$|f(w)| \le \frac{\max_C |\varphi|}{|w| - 1} \, 2\pi \cdot 1$$

$\longrightarrow 0$ as $w \to \infty$.

Type of sing of f at ∞ $\overset{\text{def}^n}{=}$ Type of sing of $f(\frac{1}{z})$ at $z = 0$.

Removable sing at ∞. In fact, f has a "zero at ∞".

f analytic on $\{z : |z| > R\}$. ∞ is an "isolated sing" on $\hat{\mathbb{C}}$. 3 types: removable, pole, essential.

2. Suppose that A is a finite set and that $f(z)$ is analytic on $\mathbb{C} - A$ with poles at each point in A. Prove that if f has a removable singularity at infinity, then f must be a rational function.

Aha! $\underset{\substack{\downarrow \\ \text{value as} \to \infty.}}{f} - \underbrace{\left(\sum_{1}^{N} \text{princ parts } A \right)}_{\to 0 \text{ at } \infty} = F$

F entire, bnd. Liouville's $\Rightarrow F \equiv c.$ ✓

Remark: Same thing true if f has a pole at ∞.

(If A infinite discrete set. $A = \{a_n\}_{n=1}^{\infty}$ and $a_n \to \infty$ as $n \to \infty$. ∞ is <u>not</u> an isolated sing.)

3. Let $a_0 = 0$ and $a_1 = 1$. The Fibonacci numbers are defined inductively via

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for } n = 0, 1, 2, \ldots$$

Find the radius of convergence of the power series $\sum a_n z^n$. Hint: Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and compare the power series for $z^2 f(z)$, $zf(z)$, and $f(z)$. Find a closed form formula for f. When you get the picture, make sure everything you say is true. (For example, don't say that $f(z)$ is defined someplace until you know it is.)

$$z^2 f(z) = \sum_{n=0}^{\infty} a_n z^{n+2} = \sum_{n=2}^{\infty} a_{n-2} z^n$$

$$z f(z) = \quad \cdots\cdots \quad = \sum_{1}^{\infty} a_{n-1} z^n$$

$$f(z) = z^2 f(z) + z f(z) \qquad \text{almost}$$

$-0-1\cdot z$

Suspect: $f(z)\left[1 - z - z^2\right] = z$

Radius of conv = dist (0, nearest pole)

To nail, have to go backwards from.

Define $f(z) = \dfrac{z}{1-z-z^2}$, get R. of C.

Show $(1-z-z^2) f(z) = z$ yields a_n = F. #s.

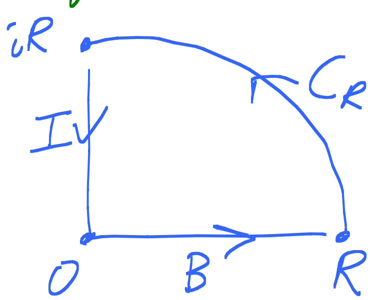
4. How many zeroes does the polynomial

$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

Arg princ: $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ zeroes } f \\ \text{inside } \gamma \text{ with mult}\end{array}\right)$

$=$ # times $f(z)$ spins around 0 as z goes around γ in C.C. sense.



Get $\int_{C_R} \frac{f'}{f}\,dz$ by letting $R \to \infty$.

Along B: $t^{1998} + t + 2001$ sticks to $\mathbb{R}$!

$\Delta_B \arg = 0$.

Along I: $(it)^{1998} + (it) + 2001$

$-t^{1998} + it \quad 2001$

$(2001 - t^{1998}) + it$ ← in UHP!

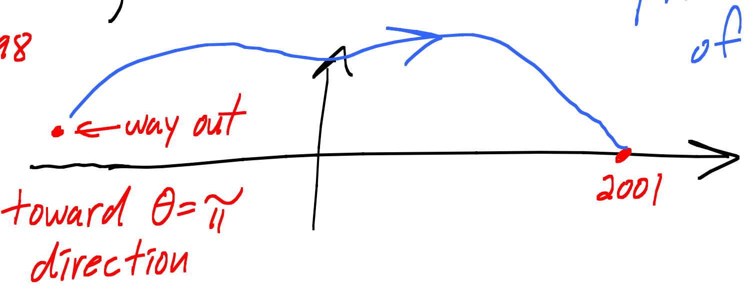
iR : (Big negative #) $+ iR$

$2001 - R^{1998}$

$i \cdot 0$: 2001

$\Delta_I \arg \to -\pi$

Can use princ branch of arg.



5. Suppose that f is a non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = e^{it}$ where $0 \le t \le 2\pi$. Suppose that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} - \{0\}$.

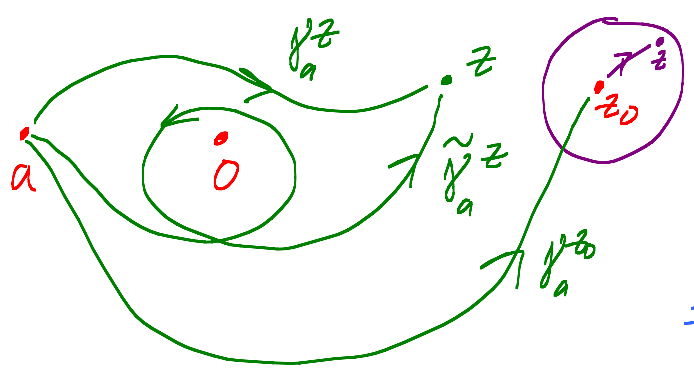
Hmmm $\quad \int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw \leftarrow ?\ \log f$

Not well defined!

Aha! "Define" $\quad F(z) = \exp\left(\frac{1}{3}\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw\right)$

Step 1 $\quad F$ well defined.

$$\frac{F(z)}{\tilde{F}(z)} = \exp\left(\frac{1}{3}\underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right)}_{\int_\gamma \ \gamma \text{ closed}} \frac{f'}{f}\,dw\right)$$



$$\frac{1}{3}\left(2\pi i \cdot N \cdot 3\right)$$

$$= e^{2\pi i N} = 1$$

Step 2 $\quad$ Is $F^3 = f$? $\quad$ Almost!

Key $\quad F$ analytic : $\quad F' = \exp\left(\frac{1}{3}\int\right)\left[0 + \frac{1}{3}\frac{f'(z)}{f(z)}\right]$

$$F' = F \cdot \frac{1}{3}\frac{f'}{f}$$

Step 3 $\quad$ Trick : $\frac{d}{dz}\left[\frac{F^3}{f}\right] \equiv 0.$

6. Suppose that $\{a_k\}_{k=1}^N$ is a finite sequence of distinct complex numbers and that f is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$. Prove that there exist constants c_j, $j = 1, 2, \ldots, N$, such that

$$f(z) - \sum_{k=1}^{N} \frac{c_k}{z - a_k}$$

has an analytic antiderivative on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$.

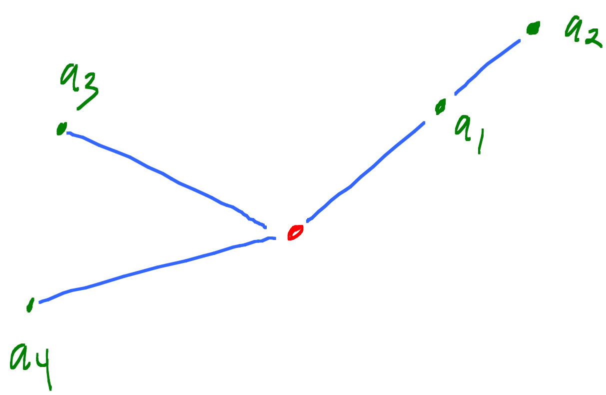
Key: analytic f has an antiderivative on a domain $\iff \int_\gamma f\,dz = 0$ for all closed γ in Ω.

Hmmm. $\int_\gamma f\,dz = 2\pi i \sum \mathrm{Ind}_\gamma(a_n)\,\mathrm{Res}_{a_n} f$

Gen Res thm.

7. Suppose $a_1, a_2, \ldots, a_N$ are distinct nonzero complex numbers and let Ω denote the domain obtained from $\mathbb{C}$ by removing each of the closed line segments joining a_k to the origin, $k = 1, \ldots, N$. Prove that there is an analytic function f on Ω such that

$$f(z)^N = \prod_{k=1}^{N}(z - a_k).$$



8. Find a one-to-one conformal mapping of the "piece of pie" $\{re^{i\theta} : 0 < r < 1,\ 0 < \theta < \pi/4\}$ onto the horizontal strip $\{z : 0 < \operatorname{Im} z < 1\}$.

9. Suppose that u is a continuous real valued function on $\overline{D_1(0)}$ and that

$$(*) \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + (1-|z|)e^{i\theta})\,d\theta$$

for each $z \in D_1(0)$. (This equality means that u is only known to satisfy the averaging property on circles like the one pictured below.) Prove that u is harmonic in $D_1(0)$.