

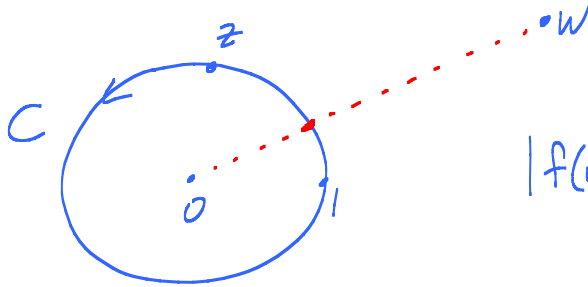
Review lecture 2

Final exam take-home exam. Available noon on Tuesday. Due 11:59 pm Wed. Open book, notes only. 3 hours max. Scan, upload in Gradescope.

- Suppose that ϕ is a continuous function on the unit circle in the complex plane, and let C denote the unit circle parametrized via $z(t) = e^{it}$, $0 \leq t \leq 2\pi$. Let $\Omega = \{w \in \mathbb{C} : |w| > 1\}$. For $w \in \Omega$, define

$$f(w) = \int_C \frac{\phi(z)}{z - w} dz.$$

What kind of singularity does f have at infinity? Use careful estimates and explain.



$$|f(w)| \leq \frac{\max_C |\phi|}{|w| - 1} 2\pi \cdot 1$$

$$\rightarrow 0 \text{ as } w \rightarrow \infty.$$

Type of sing of f at ∞ $\stackrel{\substack{\uparrow \\ \text{def}}}{=}$ Type of sing of $f(\frac{1}{z})$ at $z=0$.

Removable sing at ∞ . In fact, f has a "zero at ∞ ".

f analytic on $\{z : |z| > R\}$. ∞ is an "isolated sing" on $\hat{\mathbb{C}}$. 3 types: removable, pole, essential.

2. Suppose that A is a finite set and that $f(z)$ is analytic on $\mathbb{C} - A$ with poles at each point in A . Prove that if f has a removable singularity at infinity, then f must be a rational function.

Aha! $f - \underbrace{\left(\sum_1^N \text{princ parts } A \right)}_{\rightarrow 0 \text{ at } \infty} = F$

\downarrow
 value
 as $\rightarrow \infty$.

F entire, bnd. Liouville's $\Rightarrow F \equiv c$. ✓

Remark: Same thing true if f has a pole at ∞ .

(If A infinite discrete set. $A = \{a_n\}_{n=1}^{\infty}$ and $a_n \rightarrow \infty$ as $n \rightarrow \infty$. ∞ is not an isolated sing.)

3. Let $a_0 = 0$ and $a_1 = 1$. The Fibonacci numbers are defined inductively via

$$a_{n+2} = a_{n+1} + a_n \quad \text{for } n = 0, 1, 2, \dots$$

Find the radius of convergence of the power series $\sum a_n z^n$. Hint: Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and compare the power series for $z^2 f(z)$, $z f(z)$, and $f(z)$. Find a closed form formula for f . When you get the picture, make sure everything you say is true. (For example, don't say that $f(z)$ is defined someplace until you know it is.)

$$z^2 f(z) = \sum_{n=0}^{\infty} a_n z^{n+2} = \sum_{n=2}^{\infty} a_{n-2} z^n$$

$$z f(z) = \dots = \sum_{n=1}^{\infty} a_{n-1} z^n$$

$$f(z) = z^2 f(z) + z f(z) \quad \text{almost}$$

-0-1·z

$$\text{Suspect: } f(z) [1 - z - z^2] = z$$

Radius of conv = dist(0, nearest pole)

To nail, have to go backwards from.

Define $f(z) = \frac{z}{1-z-z^2}$, get R. of C.

Show $(1-z-z^2)f(z) = z$ yields $a_n = F_n$'s.

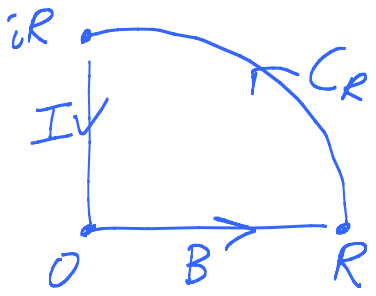
4. How many zeroes does the polynomial

$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

Arg princ: $\frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} dz = \left(\begin{array}{l} \# \text{ zeroes } f \\ \text{inside } \gamma \text{ with mult} \end{array} \right)$

$= \# \text{ times } f(z) \text{ spins around } 0 \text{ as } z \text{ goes around } \gamma \text{ in C.C. sense.}$



Get $\int_{C_R} \frac{f'}{f} dz$ by letting $R \rightarrow \infty$.

Along B: $t^{1998} + t + 2001$ sticks to \mathbb{R} !

$\Delta_B \arg = 0$.

Along I: $(it)^{1998} + (it) + 2001$
 $= -t^{1998} + it + 2001$

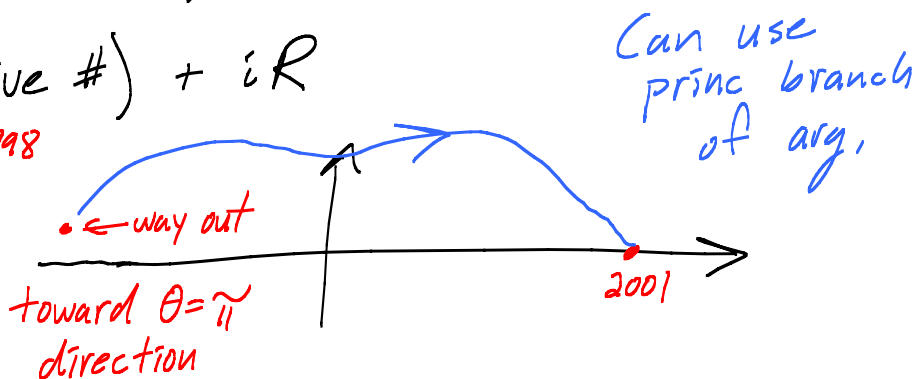
$(2001 - t^{1998}) + it \leftarrow \text{in } \underline{\underline{\text{UHP}}}$!

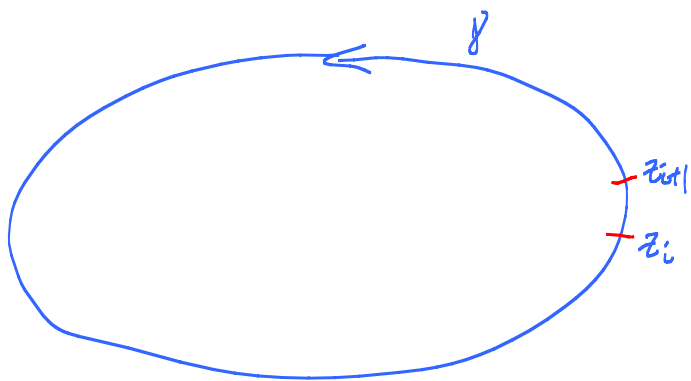
iR : (Big negative #) + iR

$\text{ArcTan } \frac{R}{2001 - R^{1998}}$

$i \cdot 0$: 2001

$\Delta_I \arg \rightarrow -\tilde{\pi}$





$$\int_{\gamma} \frac{f'}{f} dz = \sum \log f(z) \Big|_{z_i}^{z_{i+1}}$$

$$= \sum \underbrace{\ln |f(z)|}_{=0} \Big|_{z_i}^{z_{i+1}} + i \Delta_{\gamma} \operatorname{Arg} f(z)$$

Important point

$$\operatorname{Im} \left(\int_{\gamma_a^b} \frac{f'}{f} dz \right) = \Delta_{\gamma_a^b} \operatorname{Arg} f$$

$$\int_{C_R} \underbrace{\frac{1998z^{1997} - 1}{z^{1998} - z + 2001}}_{\sim \frac{1998}{z}} dz$$

$$\int_{C_R} \frac{1998}{z} dz = \int_0^{\pi/2} \frac{1998}{Re^{it}} iRe^{it} dt = i \underbrace{\frac{\pi}{2} 1998}_{\Delta_{C_R} \arg}$$

Show

$$\int_{C_R} \left(\frac{1998z^{1997} - 1}{z^{1998} - z + 2001} - \frac{1998}{z} \right) dz \xrightarrow{R \rightarrow \infty} 0$$

5. Suppose that f is a non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = e^{it}$ where $0 \leq t \leq 2\pi$. Suppose that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz$$

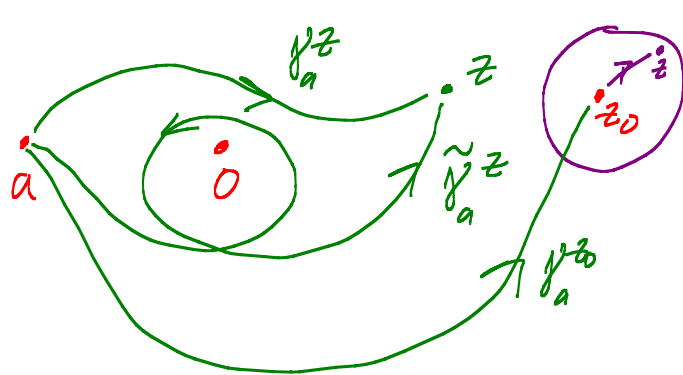
is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} - \{0\}$.

Hmmm $\int_{\gamma_a^z} \frac{f'(w)}{f(w)} dw \leftarrow ? \log f$
Not well defined!

Aha! "Define" $F(z) = \exp\left(\frac{1}{3} \int_{\gamma_a^z} \frac{f'(w)}{f(w)} dw\right)$

Step 1 F well defined.

$$\frac{F(z)}{\tilde{F}(z)} = \exp\left(\frac{1}{3} \left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z} \right) \frac{f'}{f} dw\right)$$



\int_{γ} γ closed

$$\frac{1}{3} (2\pi i \cdot N \cdot 3)$$

$$= e^{2\pi i N} = 1$$

Step 2 Is $F^3 = f$? Almost!

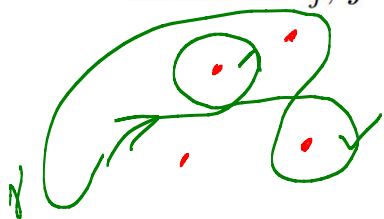
Key F analytic: $F' = \exp\left(\frac{1}{3} \int\right) \left[0 + \frac{1}{3} \frac{f'(z)}{f(z)} \right]$

$$F' = F \cdot \frac{1}{3} \frac{f'}{f}$$

Step 3 Trick: $\frac{d}{dz} \left[\frac{F^3}{f} \right] \equiv 0$. $c F^3 = f$

Ans:
 $c^{1/3} F$

6. Suppose that $\{a_k\}_{k=1}^N$ is a finite sequence of distinct complex numbers and that f is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \dots, N\}$. Prove that there exist constants c_j , $j = 1, 2, \dots, N$, such that



$$f(z) - \sum_{k=1}^N \frac{c_k}{z - a_k}$$

$$\Omega = \mathbb{C} - \{a_n\}_{n=1}^N$$

has an analytic antiderivative on $\mathbb{C} - \{a_k : k = 1, 2, \dots, N\}$.

Key: analytic f has an antiderivative on a domain $\iff \int_{\gamma} f dz = 0$ for all closed γ in Ω .

Hmmm. $\int_{\gamma} f dz = 2\pi i \sum \text{Ind}_{\gamma}(a_n) \text{Res}_{a_n} f$

\uparrow Gen Res thm. \uparrow F
 $\underbrace{\hspace{10em}}_{=0}$

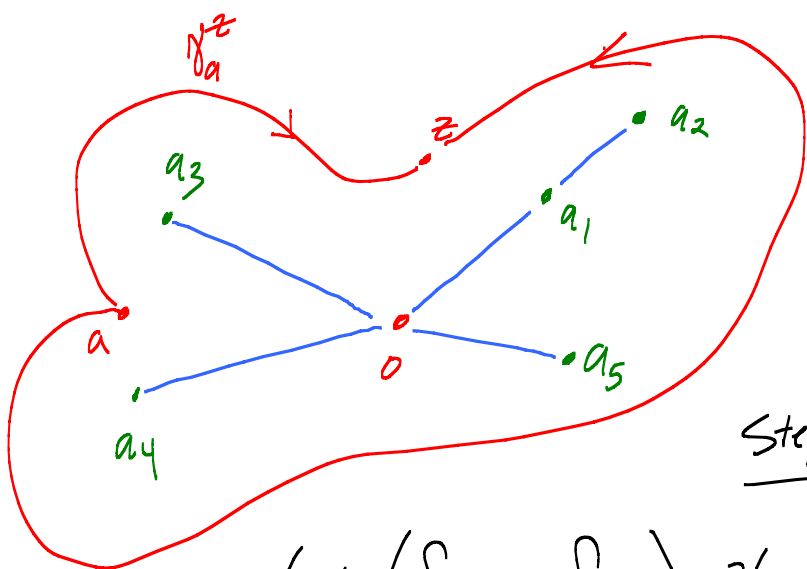
$$F = f - \sum \frac{c_k}{z - a_k}$$

$$c_k = \text{Res}_{a_k} f$$

$$\text{So } \text{Res}_{a_k} F = 0$$

7. Suppose a_1, a_2, \dots, a_N are distinct nonzero complex numbers and let Ω denote the domain obtained from \mathbb{C} by removing each of the closed line segments joining a_k to the origin, $k = 1, \dots, N$. Prove that there is an analytic function f on Ω such that

$$f(z)^N = \prod_{k=1}^N (z - a_k) = p(z)$$



Try

$$F(z) = \exp\left(\frac{1}{N} \int_{\gamma_z} \frac{p'(w)}{p(w)} dw\right)$$

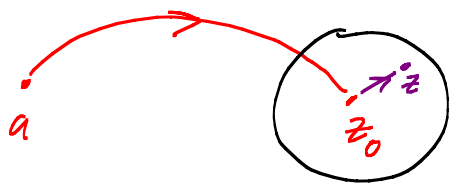
Step 1 $\frac{F(z)}{\tilde{F}(z)} =$

$$\exp\left(\frac{1}{N} \left(\int_{\gamma_z} - \int_{\tilde{\gamma}_z} \right) \frac{p'}{p} dw\right)$$

γ closed

$$= \exp\left(\frac{1}{N} 2\pi i \sum_{k=1}^N \underbrace{\text{Ind}_{\gamma}(a_k)}_{\substack{\text{all} \\ \text{same!} \\ n}} \underbrace{\text{Res}_{a_k} \frac{p'}{p}}_{\substack{= \text{order of simple zero} \\ = 1}}\right)$$

$$= \exp\left(\frac{1}{N} 2\pi i \cdot n \cdot N\right) = e^{2\pi n i} = 1 \quad \checkmark$$

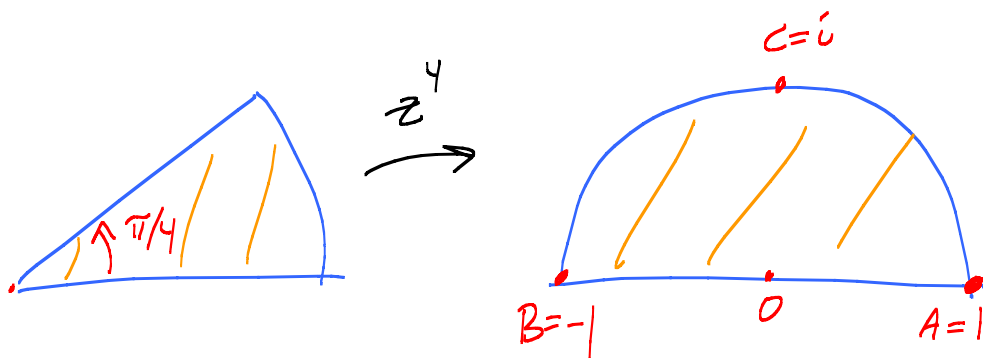


$$F' = \exp\left(\frac{1}{N} \int\right) \left[0 + \frac{1}{N} \frac{p'(z)}{p(z)}\right]$$

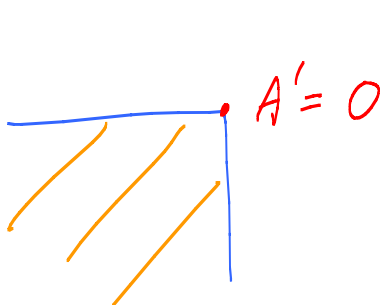
$$F' = F \frac{1}{N} \frac{p'}{p}$$

$$\frac{d}{dz} \left(\frac{F^N}{p} \right) \equiv 0, \text{ etc.}$$

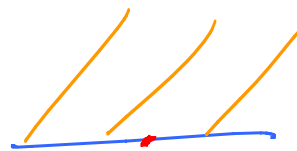
8. Find a one-to-one conformal mapping of the “piece of pie”
 $\{re^{i\theta} : 0 < r < 1, 0 < \theta < \pi/4\}$ onto the horizontal strip $\{z : 0 < \text{Im } z < 1\}$.



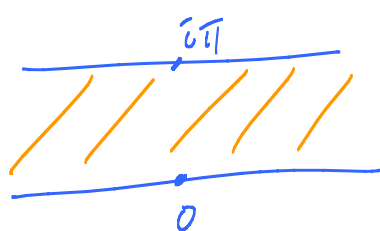
$$\frac{z-1}{z+1}$$



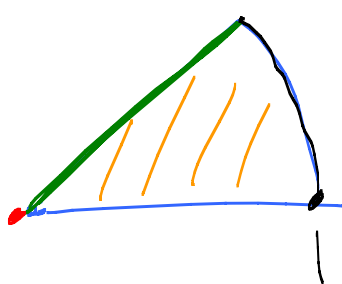
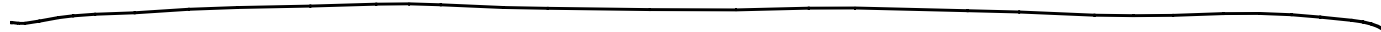
$$(-z)^2$$



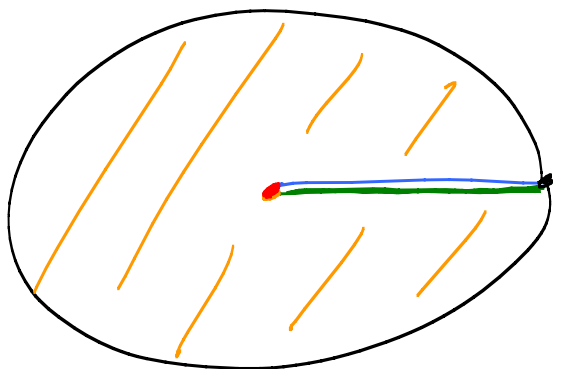
$$\text{Log } z$$



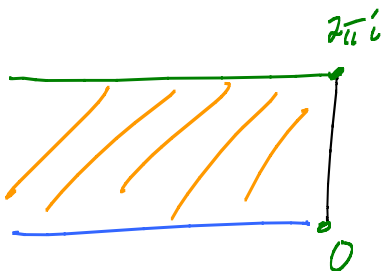
$$\frac{z}{i}$$



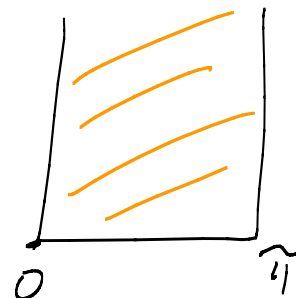
$$z^8$$



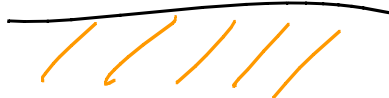
$$\log_0 z$$



$$\frac{-iz}{2}$$



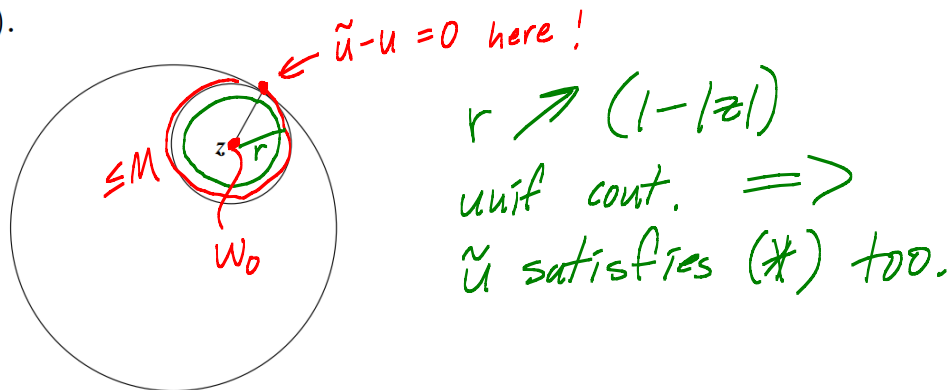
$$\cos z$$



9. Suppose that u is a continuous real valued function on $\overline{D_1(0)}$ and that

$$(*) \quad u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + (1 - |z|)e^{i\theta}) d\theta$$

for each $z \in D_1(0)$. (This equality means that u is only known to satisfy the averaging property on circles like the one pictured below.) Prove that u is harmonic in $D_1(0)$.



Let $\tilde{u} = \text{Poisson integral of } u(e^{i\theta})$.

$\tilde{u} - u \not\equiv 0$, then $\tilde{u}(z_0) - u(z_0) = c \neq 0$. Suppose $c > 0$.

$\tilde{u} - u$ has max M on $\overline{D_1(0)}$ at some pt. $w_0 \in \overline{D_1(0)}$.

$$\left(\text{Ave of } \underbrace{\tilde{u} - u}_{\substack{< M \\ \text{near edge}}} \text{ on circle} \right) = M \quad \Downarrow$$

Case $c < 0$: Apply to $u - \tilde{u}$ instead. Same \Downarrow .

So $\tilde{u} - u \equiv 0$. $u = \tilde{u}$ is harmonic.