Lecture 2 The Cauchy-Riemann equations (C-R equns)

$\lim_{z \to z_0} f(z) = L$: Given $\varepsilon > 0$, $\exists\, \delta > 0$ ...

$$|f(z) - L| < \varepsilon \quad \text{when} \quad |z - z_0| < \delta, \; z_0 \neq z.$$

Real analysis book : Change x's to z's, $|\ \cdots\ |$ stands for modulus. Get complex version.

Fact: Basic facts of $\mathbb{R}$ calc $\rightsquigarrow$ $\mathbb{C}$ calc.

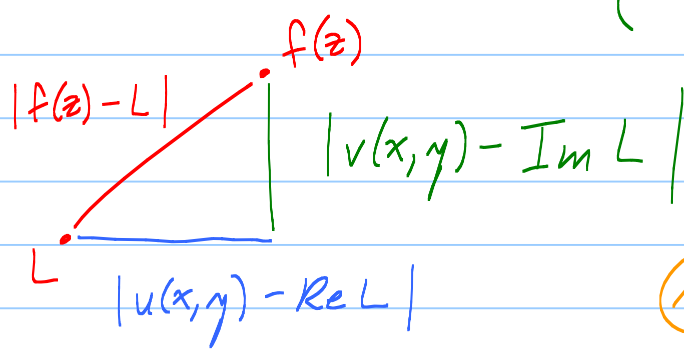
Ex: $(fg)' = fg' + f'g$ etc.

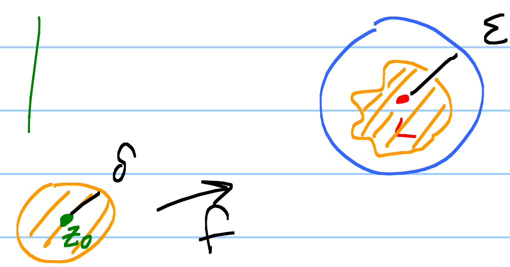
Chain rule too!

Little lemma: $f(x+iy) = u(x,y) + iv(x,y)$

$$\lim_{z \to z_0} f(z) = L \iff \begin{cases} \lim_{(x,y) \to (x_0,y_0)} u(x,y) = \operatorname{Re} L \\ \quad \cdots\cdots \quad v \ldots = \operatorname{Im} L \end{cases}$$

Pf:





Def^n: f is $\mathbb{C}$-diff'ble at a :

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \text{ exists. Call it } f'(a).$$

Ex: $f(z) = z^2$ is $\mathbb{C}$-diff.

$$DQ = \frac{z^2 - a^2}{z - a} = z + a \longrightarrow 2a \quad \text{as } z \to a.$$

EX: $f(z) = \bar{z}$ $\qquad z = x+iy, \quad \bar{z} = x - iy$

is not $\mathbb{C}$-diff!

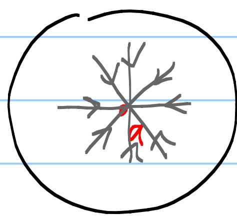
Facts: $|zw| = |z|\,|w|$ $\qquad \overline{(zw)} = \bar{z}\,\bar{w}$

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}, \quad w \neq 0$ $\qquad \overline{(z/w)} = \bar{z}/\bar{w}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta = \overline{(e^{i\theta})} = \frac{1}{e^{i\theta}} \quad (N=-1, \text{ DeMoivre's})$$

$$DQ = \frac{\bar{z} - \bar{a}}{z - a} \qquad re^{i\theta} = z - a$$

$$= \frac{\overline{(re^{i\theta})}}{re^{i\theta}} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-i\theta}\underbrace{\left(\frac{1}{e^{i\theta}}\right)}_{e^{-i\theta}} = e^{-i2\theta}$$



Different limits along rays!

Def^n: (Open set Ω). f is analytic on Ω (or holomorphic) if it is $\mathbb{C}$-diff'ble at every pt in Ω.

Thm: $\mathbb{C}$-diff'ble fcn at a is cont. at a.

Pf: $\frac{f(z) - f(a)}{z - a} = f'(a) + E(z)$ where $E(z) \to 0$ as $z \to a$.

$$f(z) - f(a) = (z-a)f'(a) + (z-a)E(z)$$
$$0 \cdot f'(a) + 0 \cdot 0 = 0 \quad \text{as } z \to a$$

So $\lim_{z \to a} f(z) = f(a)$ ✓

<u>Cauchy – Riemann eqns</u> If $f(x+iy) = u(x,y) + iv(x,y)$ is $\mathbb{C}$-diff'ble at $z_0 = x_0 + iy_0$, then the first partial deriv's of u and v <u>exist</u> at (x_0, y_0) and satisfy

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \qquad \text{C-R eqns}$$

<u>Notation</u>: $u_x = \dfrac{\partial u}{\partial x}$, etc. $\qquad u^0 = u(x_0, y_0)$, etc.

$\qquad u_x^0 = u_x(x_0, y_0)$, etc.

<u>Pf</u>: <u>Step 1</u> Let $z \to z_0$ along horiz

$$DQ = \frac{[u(x,y_0) + i\,v(x,y_0)] - [u^0 + iv^0]}{(x+iy_0) - (x_0 + iy_0)}$$

$$= \underbrace{\frac{u(x,y_0) - u(x_0,y_0)}{x - x_0}}_{\downarrow\ \operatorname{Re} f'(z_0)} + i\,\underbrace{\frac{v(x,y_0) - v^0}{x - x_0}}_{\downarrow\ \operatorname{Im} f'(z_0)}$$

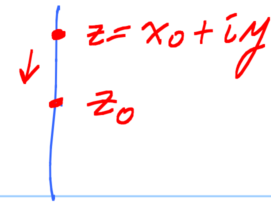
Baby lemma

Conclude: Partials exist and

$$\boxed{f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0)}$$

Red box #1

Step 2 slide vertical



$$DQ = \frac{\ell\ell\ell\ell}{(x_0+iy)-(x_0+iy_0)} = \frac{\ell\ell\ell\ell}{i(y-y_0)}$$

$$= \underbrace{\frac{1}{i}}_{-i}\left[\frac{u(x_0,y)-u^0}{y-y_0} + i\,\frac{v(x_0,y)-v^0}{y-y_0}\right]$$

$$\underbrace{v_y^0}_{\mathrm{Re}\, f'(z_0)} + \underbrace{-i\,u_y^0}_{i\,\mathrm{Im}\, f'(z_0)}$$

Conclude: partials exist and

$$\boxed{f'(z_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)}$$ Red box #2

Equate red boxes to get C-R eqns.

Ex: $\bar{z} = x - iy$ C-R eqns fail!

Ex: $E(x+iy) = e^x \cos y + i e^x \sin y$ $\quad (e^z)$

C-R eqns hold. Is it analytic?

Partial converse to C-R equns: (Ω open set)

If u, v are C^1-smooth on Ω and satisfy the C-R equns, then $f(x+iy) = u(x,y) + iv(x,y)$ is analytic on Ω.

Looman-Menchoff thm: 1) u, v cont on Ω
2) first partials exist
3) satisfy C-R equns

Then $u + iv$ is analytic on Ω.

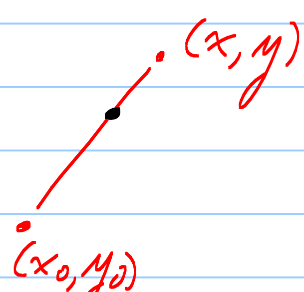
Pf: Narasimhan: Complex anal in 1 variable p. 43-50.

Calculus fact: Suppose u is C^1-smooth.

$$u(x,y) = u^0 + u_x^0 \cdot (x - x_0) + u_y^0 \cdot (y - y_0) + R(x,y)$$

where
$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Pf: $$u(x,y) - u^0 = \int_0^1 \frac{\partial}{\partial t} u\big(x_0 + t(x-x_0),\, y_0 + t(y-y_0)\big)\, dt$$



$$= \int_0^1 u_x(\quad)\cdot(x-x_0) + u_y(\quad)\cdot(y-y_0)\ dt$$

Now subtract $\underbrace{u_x^0\cdot(x-x_0)+u_y^0\cdot(y-y_0)}_{A=\int_0^1 A\,dt}$

Get $u-$ (first order Taylor poly)

$$R(x,y)=\int_0^1 \underbrace{\left[u_x(\ )-u_x^0\right]}_{\text{small}}(x-x_0)+\underbrace{\left[u_y(\ )-u_y^0\right]}_{\text{small}}(y-y_0)\,dt$$

$$\left|\frac{R(x,y)}{\sqrt{}}\right|\le \dots\underbrace{\frac{|x-x_0|}{\sqrt{}}}_{\le 1}+\dots\underbrace{\frac{|y-y_0|}{\sqrt{}}}_{\le 1}\dots$$

$\le 2\varepsilon$ if $\operatorname{dist}((x,y),(x_0,y_0))<\delta$.