Lecture 3 C-R equs and the complex exponential $E(z)=e^z$

Last time: u $\mathcal{C}^1$-smooth:

$$u(x,y) = \underbrace{u(x_0,y_0)}_{u^0} + \underbrace{u_x(x_0,y_0)}_{u_x^0}(x-x_0) + \underbrace{u_y(x_0,y_0)}_{u_y^0}(y-y_0) + R_u(x,y)$$

where $\dfrac{R_u(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \longrightarrow 0$ as $(x,y)\to(x_0,y_0)$

$\sqrt{} \to |z-z_0|$ $\quad z=x+iy \quad z_0=x_0+iy_0$

Thm: u,v $\mathcal{C}^1$-smooth on Ω and satisfy C-R equs, then

$$f(x+iy) = u(x,y)+iv(x,y) \text{ is } \underline{\text{analytic}}$$

(on open set Ω).

Pf: $DQ = \dfrac{f(z)-f(z_0)}{z-z_0} = \dfrac{(u+iv)-(u^0+iv^0)}{z-z_0}$

$$= \frac{\left[u_x^0\cdot(x-x_0)+\overset{-v_x^0}{u_y^0}\cdot(y-y_0)\right]+i\left[v_x^0\cdot(x-x_0)+\overset{=u_x^0}{v_y^0}\cdot(y-y_0)\right]}{z-z_0} + \underbrace{\frac{R_u+iR_v}{z-z_0}}_{\varepsilon}$$

$$= \frac{[u_x^0+iv_x^0]\cdot[(x-x_0)+i(y-y_0)]}{z-z_0} + \varepsilon$$

$$= \overset{\#1}{\boxed{u_x^0+iv_x^0}} + \varepsilon$$

where $|\varepsilon| \le \dfrac{|R_u|}{|z-z_0|} + \dfrac{|R_v|}{|z-z_0|} \longrightarrow 0$ as $z \mapsto z_0$.

Cor: $E(x+iy) = \underbrace{e^x\cos y}_{u} + i\underbrace{e^x\sin y}_{v}$ is analytic on $\mathbb{C}$

e^z entire fcn

Properties of $E(z)$: 0) $E(z)$ is entire

1) $E(0)=1$

2) $E'(z) = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} = u + iv = E(z)$

Fun thing: 1-3 determine $E(z)$:

$$\begin{cases} u_x = u & v_x = v \\ v_y = u & -u_y = v \end{cases}$$

$u(0,0)=1, \quad v(0,0)=0.$

e.g. $u_x = u$. $\quad u = C e^x$ $\quad C = C(y)$

$v = S(y)e^x$

Get $u_{yy} + u = 0$, $v_{yy} + v = 0$. Initial cond, etc.

3) $E(z+w) = E(z)\cdot E(w)$

Grp homomorphism: (add grp) $\to$ (mult grp)

4) $E(z)E(-z) = 1$ so $E(-z) = 1/E(z)$

5) $E(z)$ never zero. $\quad$ Pf. (3): $w=-z$.

6) $E(z-w) = E(z)/E(w)$

7) $E(x) = e^x$, $x \in \mathbb{R}$. $\quad E(z)$ extends e^x to $\mathbb{C}$.

8) $E(i\theta) = \cos\theta + i\sin\theta$

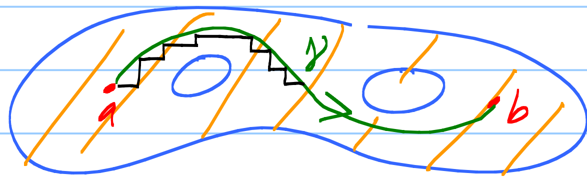
#3 $\Rightarrow$ DeMoivre's $N>0$. #4: $N<0$.

9) $E(z)=1 \iff z=2\pi n i,\ n\in\mathbb{Z}.$

Just need to prove #3.

Lemma 1 If f analytic on $D_r(a)$ (or $\mathbb{C}$) and $f'\equiv 0$, then f is const.

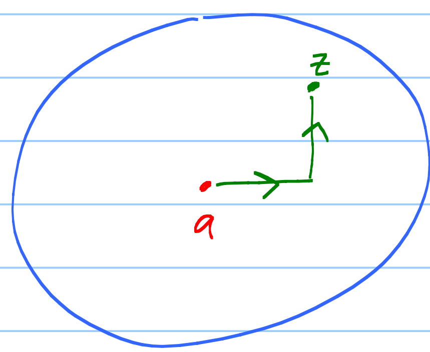
Wish pf: $f' = \begin{cases} u_x + iv_x \\ v_y - iu_y \end{cases} \equiv 0 \implies \begin{matrix} \nabla u \equiv 0 \\ \nabla v \equiv 0 \end{matrix}$



$$\int_\gamma \underbrace{\nabla u}_{\equiv 0} \cdot d\vec{s} = \underbrace{u(b) - u(a)}_{0}$$

Ouch! Need u C^1-smooth

Back to pf:



Fresh calc: $h(t)$ is cont on $[a,b]$ and diff'ble on (a,b) and $h'(t)\equiv 0$ on (a,b). MVT $\implies h \equiv$ const on $[a,b]$.

Fancier thm: $f'\equiv 0$ on open connected set $\implies f\equiv$ const on set.

Lemma 2: If f is analytic on $D_R(0)$ (or $\mathbb{C}$) and $f'(z) = k f(z)$ there ($k \in \mathbb{C}$ is a const), then

$f(z) = C E(kz)$ where $C = f(0)$.

Pf: $f' - kf \equiv 0$ Int. factor $E(-kz)$

$$E(-kz)\left[f' - kf\right] \equiv 0$$

$$\underbrace{E(-kz) f' - k E(-kz) f}_{\frac{d}{dz}\left[E(-kz) f(z)\right]} \equiv 0$$

Chain rule:
$\frac{d}{dz} E(-kz) = \underbrace{E'(-kz)}_{E(-kz)} \underbrace{\frac{d}{dz}(-kz)}_{-k}$
$-k E(-kz)$

Lemma 1 $\Rightarrow$ $\boxed{E(-kz) f(z) = C}$

Stop the proof! Start over and let $f(z) = E(kz)$.

Note: $f'(z) = k E(kz) = k f(z)$.

So this f satisfies hyp and green box:

$E(-kz) E(kz) = C$. $z = 0 \Rightarrow C = 1$.

Take $k = 1$. $\boxed{E(-z) E(z) = 1}$ $E(-kz) = 1/E(kz)$.

Restart pf with random f.

Green box $\underbrace{E(-kz)}_{1/E(kz)} f(z) = C$ $z = 0$: $C = f(0)$

$f(z) = f(0) E(kz)$ ✓

Pf of #3: Let $f(z) = E(z+w)$, w const.

$$f'(z) = \underbrace{E'(z+w)}_{E(z+w)} \cdot \underbrace{\frac{d}{dz}[z+w]}_{1}$$

$$f'(z) = f(z) \qquad (k=1)$$

Lemma 2 $\Longrightarrow$ $f(z) = f(0)\, E(1 \cdot z)$

$$E(z+w) = E(w) \cdot E(z) \quad \checkmark$$

Fun thing: Now can deduce all the trig idendities!

Big fact f analytic $\Longrightarrow$ f' analytic $\Longrightarrow$ f'' analytic ...

$\Longleftarrow$ f' continuous

So all $\mathbb{C}$-deriv exist and are cont!

Way false $\mathbb{R} \to \mathbb{R}$!

All red boxes for all deriv's $\Longrightarrow$ u, v are C^∞-smooth!

Taylor: $\begin{cases} \text{analytic: } \mathbb{C}\text{-diff'ble on open } \Omega \\ \text{holomorphic: } f' \text{ exists and is cont on } \Omega. \end{cases}$

Today: holo $\Longrightarrow$ analytic $\checkmark$

Soon: $\Longleftarrow$.

Big fact: $u+iv$ analytic. Then u, v are C^∞-smooth harmonic fcns! $\begin{cases} \Delta u \equiv 0 \\ \Delta v \equiv 0 \end{cases}$