

Lecture 4 Complex power series

Big fact: f analytic $\Rightarrow f'$ analytic

$\Rightarrow u, v$ are C^∞ -smooth.

Assuming big fact (or just that u and v are C^2 -smooth),

then $f = u + iv$ analytic $\Rightarrow u, v$ harmonic!

Why: C-R eqns

$$\begin{cases} u_x = v_y & (A) \\ u_y = -v_x & (B) \end{cases}$$

$$\begin{aligned} & \frac{\partial^2}{\partial x^2}(A): u_{xx} = v_{xy} \\ + & \frac{\partial^2}{\partial y^2}(B): u_{yy} = -v_{yx} \end{aligned}$$

$$v \text{ } C^2 \text{ smooth} \Rightarrow v_{xy} = v_{yx}$$

$$\Delta u = u_{xx} + u_{yy} = 0 \quad u \text{ harmonic!}$$

Similarly, $\frac{\partial^2}{\partial y^2}(A) - \frac{\partial^2}{\partial x^2}(B)$ shows that v is harmonic.

Big fact: u C^2 -smooth and harmonic $\Rightarrow u$ C^∞ -smooth.

Problem: Given harmonic u , does there exist harmonic v such that $u + iv$ is analytic? [v is called a

harmonic conjugate for u .]

Ans: Yes, when Ω has no holes.

How to cook up harmonic conjugate v for harmonic C^2 harmonic fcn u :

Want v so that $u+iv$ is analytic: C-R eqns $\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix} \rightarrow \begin{matrix} v_x = -u_y \\ v_y = u_x \end{matrix}$

Prob: Find v with $\nabla v = \vec{F}$

$\uparrow$ potential fcn for $\vec{F}$

in $\mathbb{R}^3$: $\Leftrightarrow \text{Curl } \vec{F} \equiv 0$

in $\mathbb{R}^2$: $\Leftrightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

or Ω with
no holes

$\leftarrow (v_{yx} = v_{xy})$

$\nabla v = \vec{F}$
 (v_x, v_y)

$\nabla v = \vec{F}$
where $\vec{F} = (-u_y, u_x)$

Shown in MA 261.

Check:

$$\frac{\partial}{\partial y} \underbrace{(-u_y)}_{F_1} \stackrel{?}{=} \frac{\partial}{\partial x} \underbrace{(u_x)}_{F_2}$$

Yes! $u_{yy} + u_{xx} \equiv 0$ ✓

Fact: v is unique up to $+C$.

Power series:

Lem: $\sum_1^\infty a_n$ conv $\Rightarrow a_n \rightarrow 0$.

PF: $S_N = \sum_1^N a_n$ $a_N = S_N - S_{N-1} \rightarrow L - L$ as $N \rightarrow \infty$

Defⁿ: $\sum_1^\infty a_n$ conv absolutely if $\sum_1^\infty |a_n| < \infty$.

Lemma: abs conv $\Rightarrow$ conv.

PF: Abs conv $\Rightarrow S_N$ Cauchy seq $\Rightarrow \sum_1^\infty a_n$ conv.

Big Fact: Power series have a radius of conv. R , meaning

1) $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ conv. abs. in $D_R(z_0)$

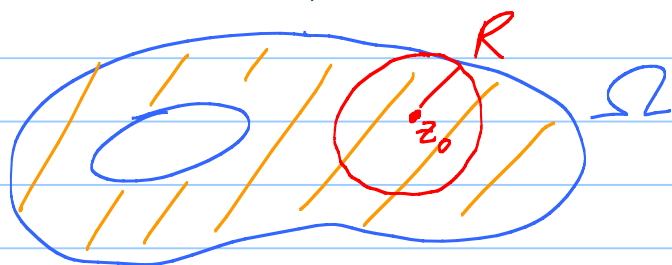
2) conv uniformly on $D_r(z_0)$, $r < R$.

3) div if $|z-z_0| > R$.

$R=0$ case: only conv at z_0 .

$R=\infty$ case: conv for all $z \in \mathbb{C}$.

Big Fact: Analytic fns are given locally by conv power series with $R > 0$.



Pf: later

Hadamard's Formula:

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

Defⁿ: $\limsup_{n \rightarrow \infty} r_n = \lim_{N \rightarrow \infty} \underbrace{\sup \{r_n : n \geq N\}}_{\text{non-increasing in } N}$

So limit exists as $N \rightarrow \infty$.

(Might be $-\infty$.)

(But, if $r_n = \sqrt[n]{|a_n|}$, $\limsup \geq 0$.)

Later: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Hadamard's: $\frac{1}{R} = \limsup_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n!}}$

Stirling's formula:

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{4n}\right)$$

$$\frac{1}{\sqrt[n]{n!}} < \frac{1}{(\sqrt{2\pi n})^{1/2n} \left(\frac{n}{e}\right)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So $\frac{1}{R} = 0$ and $R = \infty$.

Ratio test: much easier!

Geometric series:

$$\sum_{n=0}^{\infty} z^n$$

$|z| \geq 1$
terms $\nrightarrow 0$.
So div.

$$\begin{aligned} S_N &= 1 + z + \dots + z^N \\ z S_N &= z + \dots + z^N + z^{N+1} \\ \hline (1-z) S_N &= 1 - z^{N+1} \end{aligned}$$

$$S_N = \frac{1}{1-z} + \underbrace{\frac{-z^{N+1}}{1-z}}_{E_N(z)}$$

Δ ineq: $|z+w| \leq |z| + |w|$ $\leftarrow$ Numerator estimate

$|z-w| \geq ||z| - |w||$ $\leftarrow$ Denom est

$|z+w| \geq ||z| - |w||$ (replace w by $-w$)

IP $|z| < r < 1$, then

$$|1-z| \geq |1-|z|| = 1-|z| > 1-r.$$

$$\text{So } |E_N(z)| = \frac{|z|^{N+1}}{|1-z|} \leq \frac{|z|^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-r} \rightarrow 0 \text{ as } N \rightarrow \infty$$

see $\sum z^n$ conv see unif conv.

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Note: $E_N(x) = \frac{-x^{N+1}}{1-x} \rightarrow -\infty$ as $x \nearrow 1$.

Do not get unif conv on $D_1(0)$.

PF of Hadamard's. Case $0 < R < \infty$. Assume $z_0 = 0$.

Pick ρ, r with $|z| < r < \rho < R = \text{Hadamard's}$

$$\frac{1}{R} < \frac{1}{\rho}$$

$\leftarrow \sup_{n \geq N} \sqrt[n]{|a_n|}$
sliding down to $\frac{1}{\rho}$

So, $\exists N$ such that $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ when $n \geq N$.

$$|a_n| < \frac{1}{\rho^n} \quad \text{when } n \geq N$$

$$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \quad \text{when } n \geq N.$$

$\uparrow < 1$

Aha! Comparison with geom series $\sum (\frac{|z|}{\rho})^n$ shows $\sum_{n=0}^{\infty} a_n z^n$ is

abs. conv., Can make r as close to R as desired.

So $\sum a_n z^n$ conv abs in $D_R(0)$.

Next: Get unif conv on $D_r(0)$. $f(z) = \sum_{n=0}^{\infty} a_n z^n$

$$\left| f(z) - \sum_{n=0}^M a_n z^n \right| = \left| \sum_{n=M+1}^{\infty} a_n z^n \right|$$

$$\leq \sum_{n=M+1}^{\infty} |a_n z^n| \quad \text{and if } M > N:$$

$$\leq \sum_{n=M+1}^{\infty} \left(\frac{|z|}{\rho}\right)^n = \frac{\left(\frac{|z|}{\rho}\right)^{M+1}}{1 - \left(\frac{|z|}{\rho}\right)}$$

$$< \frac{\left(\frac{r}{\rho}\right)^{M+1}}{1 - \frac{r}{\rho}}$$

$\rightarrow 0$ as $M \rightarrow \infty$.

Get unif conv on $D_r(0)$.

Last step: Div if $|z| > R$.

$$\frac{1}{|z|} < \frac{1}{R} \leq \sup_{n \geq N} \sqrt[n]{|a_n|}$$

So, for each N , $\exists n$ with $n \geq N$ such that

$$\frac{1}{|z|} < \sqrt[n]{|a_n|}$$

$$1 \leq \sqrt[n]{|a_n|} |z| \quad \leftarrow \text{raise to } n\text{-th power}$$

$$1 \leq |a_n z^n|$$

Aha! This shows that terms $a_n z^n \not\rightarrow 0$.

So series diverges.

$R=0, \infty$ cases are similar and easier.

EX: $\sum_{n=0}^{\infty} n! z^n$, $R=0$

EX: $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$, $R=1$ Converges uniformly on $\overline{D_1(0)}$

$\{z: |z| \leq 1\}$
closed unit
disc.