

Lecture 5

Complex integration

Ratio test Suppose $\lim_{n \rightarrow \infty} \left| \frac{z_{n+1}}{z_n} \right|$ exists = L

$L < 1$: $\sum_1^\infty z_n$ converges. $\leftarrow$ compare tail end of series with convergent geom series

$L > 1$: $\sum_1^\infty z_n$ diverges $\leftarrow$ terms $\nrightarrow 0$

$L = 1$: test fails : e.g. $\sum \frac{1}{n^2}$ conv, $\sum \frac{1}{n}$ div

Remark: Suppose $\sum_0^\infty a_n z^n$ has R.o.C. R . Since $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$,

Hadamard's $\Rightarrow \sum_{n=1}^\infty n a_n z^{n-1}$ has same R.o.C. R .

Big fact: Power series converge to analytic fns inside $D_R(z_0)$,

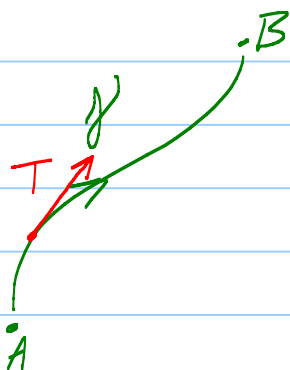
$R = R.o.C.$, and can diffⁿ term by term,

R.o.C. of $f' = R.o.C.$ of f .

Bigger fact: Analytic fns are given locally by conv. power series.

PF's : Later

Integration along paths



$$\gamma: z(t) = x(t) + iy(t) \\ a \leq t \leq b$$

$$z: [a, b] \rightarrow \mathbb{C}$$

Defⁿ: $\text{tr}(\gamma) = \text{"trace of } \gamma" = \{z(t) : t \in [a, b]\}$

$$z'(t_0) = \lim_{t \rightarrow t_0} \frac{z(t) - z(t_0)}{t - t_0} = \lim (DQ \cdot x) + i (DQ \cdot y)$$

$$= x'(t_0) + i y'(t_0)$$

Complex tangent vector $T = z'(t)$

Unit complex tang vect $\hat{T} = z'(t) / |z'(t)|$

Chain rule #1

$g: \Omega_1 \rightarrow \Omega_2$ analytic

$f: \Omega_2 \rightarrow \mathbb{C}$ analytic

$h(z) = f(g(z))$ is analytic and $h'(z) = f'(g(z)) g'(z)$

Pf: Hwk 1 or Baby Rudin.

Chain rule #2: f analytic on an open set containing

$\text{tr}(\gamma)$ and $z(t)$ is diffⁿble, then

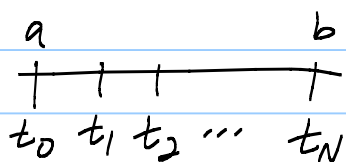
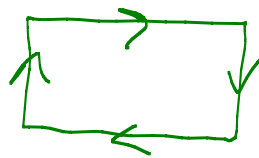
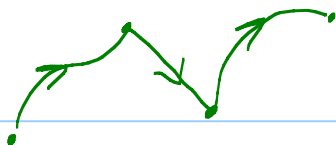
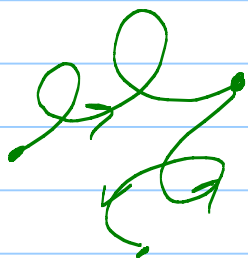
$$\frac{d}{dt} f(z(t)) = f'(z(t)) z'(t)$$

Pf: $f(z(t)) = u(x(t), y(t)) + i v(x(t), y(t))$

$$\frac{d}{dt} f(z(t)) = \left(u_x \dot{x} + \overset{-v_x}{u_y} \dot{y} \right) + i \left(v_x \dot{x} + \overset{u_x}{v_y} \dot{y} \right)$$

$$= \underbrace{[u_x + i v_x]}_{\text{red box \#1}} \cdot (\dot{x} + i \dot{y}) \quad \checkmark$$

Piecewise C^1 paths



$z(t)$ cont on $[a, b]$,
 $z(t)$ continuously diff'ble on (t_n, t_{n+1}) ,
 $z'(t)$ extends continuously to $[t_n, t_{n+1}]$

Defⁿ: $\int_a^b z(t) dt = \lim_{\Delta t \rightarrow 0} \sum_{n=1}^N z(t_n) \Delta t_n$

$$= \int_a^b x(t) dt + i \int_a^b y(t) dt$$

EX: $\int_0^{2\pi} e^{it} dt = \int_0^{2\pi} \cos t dt + i \int_0^{2\pi} \sin t dt = 0$

Fund Thm Calc #1: $\int_a^b z'(t) dt = z(b) - z(a)$
when $z(t)$ is continuously diff'ble.

True for piecewise C^1 $z(t)$ too:

$$\int_a^b z'(t) dt = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} z'(t) dt = z(t_N) - z(t_0)$$

$\underbrace{z(t_{n+1}) - z(t_n)}_{\text{"telescopes"}}$

Pf real and imag parts. Freshman calc.

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Lemma 9: $\left| \int_a^b z(t) dt \right| \leq \int_a^b |z(t)| dt$

Think: $\left| \sum z(t_n) \Delta t_n \right| \leq \sum |z(t_n)| \Delta t_n$

Let $\Delta t \rightarrow 0$.

Pf: $\int_a^b z(t) dt = r e^{i\theta}$ $\lambda = e^{-i\theta}$

$$\left| \int_a^b z(t) dt \right| = r = \lambda \int_a^b z(t) dt$$

$$= \int_a^b \lambda z(t) dt$$

check

$$= \int_a^b \underbrace{\operatorname{Re} \lambda z(t)}_{\leq |\lambda z(t)|} dt + i \int_a^b \underbrace{\operatorname{Im} \lambda z(t)}_{\text{must be zero}} dt$$

$\underbrace{|\lambda z(t)|}_{|\lambda| |z(t)|}$

$$\leq \int_a^b |z(t)| dt \quad \checkmark$$

Defⁿ: Suppose $f: \operatorname{tr}(\gamma) \rightarrow \mathbb{C}$ is continuous.

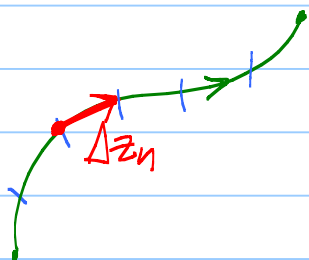
$$\int_{\gamma} f dz = \int_a^b f(z(t)) \underbrace{z'(t) dt}_{dz}$$

$\uparrow$
defⁿ

Konrad Knopf

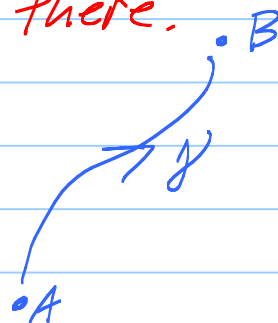
$$\int_{\gamma} f dz = \lim_{\Delta t \rightarrow 0} \sum f(z(t_n)) \underbrace{\Delta z_n}_{z(t_{n+1}) - z(t_n)}$$

$$\approx z'(t_n) (t_{n+1} - t_n)$$



Fund Thm Calc #2: f analytic on an open set containing γ and f' is continuous there.

$$\text{Then } \int_{\gamma} f' dz = f(B) - f(A)$$



Note: Later, we will show that

$$f \text{ analytic} \Rightarrow f' \text{ analytic} \Rightarrow f'' \text{ analytic} \Rightarrow \dots$$

$$\Rightarrow f' \text{ continuous}$$

Don't need red stuff.

$$\begin{aligned} \text{Pf: } \int_{\gamma} f' dz &= \int_a^b \underbrace{f'(z(t)) z'(t)}_{= \frac{d}{dt} f(z(t))} dt \\ &\quad \text{Chain rule \#2} \end{aligned}$$

$$= f(z(b)) - f(z(a)) \quad \text{Fund Thm Calc \#1}$$



Lemma (Basic estimate)

$$\left| \int_{\gamma} f \, dz \right| \leq \left(\max_{z \in \text{tr}(\gamma)} |f(z)| \right) \cdot \text{Length}(\gamma)$$

$$\text{where } \text{Length}(\gamma) = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2} \, dt = \int_a^b |z'(t)| \, dt.$$

$$\begin{aligned} \text{Pf: } \left| \int_{\gamma} f \, dz \right| &= \left| \int_a^b f(z(t)) z'(t) \, dt \right| \\ &\leq \int_a^b \underbrace{|f(z(t))|}_{\leq M = \max_{\gamma} |f|} |z'(t)| \, dt \end{aligned}$$

$$\leq M \underbrace{\int_a^b |z'(t)| \, dt}_{= L}$$

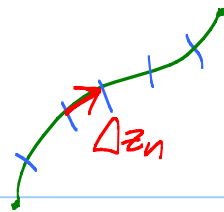
$$\text{Fact: } \gamma: z(t) \quad a \leq t \leq b$$

$$-\gamma: z(b + t(a-b)) \quad 0 \leq t \leq 1$$

$$\boxed{\int_{-\gamma} f \, dz = - \int_{\gamma} f \, dz}$$

Pf: Write out. Equate real and imag parts.
Use real variables change of var formula.

Or; Konrad Knopf

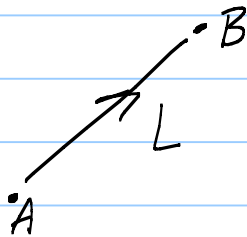


For -1 , $\Delta z_n = -\Delta z_n$.

Note: Assume $z'(t) \neq 0$ so we don't stop.

Fact: $\int$'s are independent of the parameterization.

EX:

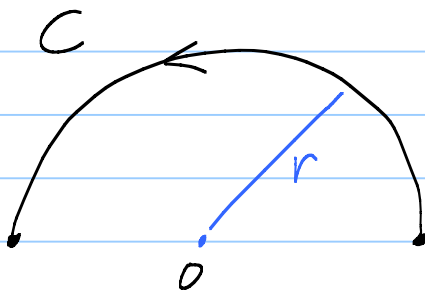


$$L: z(t) = A + t(B-A)$$

$$0 \leq t \leq 1$$

$$z'(t) = B - A$$

EX:



$$C: z(t) = re^{it}$$

$$0 \leq t \leq \pi$$

$$z'(t) = ire^{it}$$

$$\frac{d}{dt} E(it) = \underbrace{E'(it)}_{E(it)} \underbrace{\frac{d}{dt}(it)}_i$$



EX: $\int_C z^n dz, \quad n \neq -1$

$\frac{d}{dz} \left[\frac{z^{n+1}}{n+1} \right]$

Use Fund Thm Calc #2.