Lecture 6 Goursat's Lemma

HWK 2 due in GS Thurs 11:59 pm

$n \in \mathbb{Z}, \neq 1$

Important fact: γ closed piecewise C^1 path, then

$$\int_\gamma z^n\,dz = \int_\gamma \frac{d}{dz}\left[\frac{1}{n+1} z^{n+1}\right] dz = \left[\frac{1}{n+1} z^{n+1}\right]_{\text{START}}^{\text{END}} = 0$$

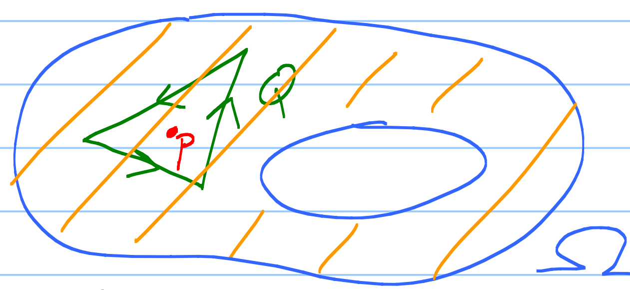
Fund Thm Calc #2

Consequently, $\int_\gamma P(z)\,dz = 0$, γ closed, P complex poly

In particular, $\int_\gamma Az + B\,dz = 0$.

Goursat's Lemma $\Omega \subset \mathbb{C}$ open, $p \in \Omega$, $f : \Omega \to \mathbb{C}$ continuous on Ω (including at p) and analytic on $\Omega - \{p\}$, Q is a Δ in Ω (and $\overline{\Delta} \subset \Omega$).

Then $\int_Q f\,dz = 0$.



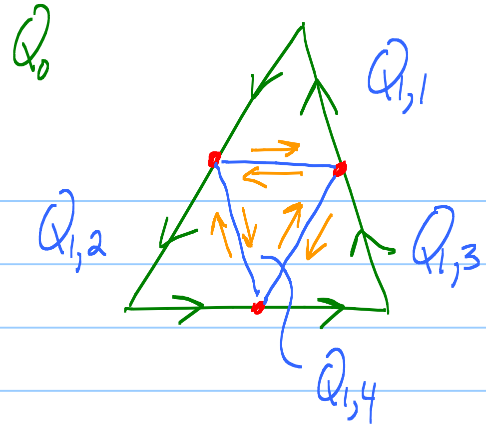
Abuse of notation: Q = open triangle Δ

Q = Δ piecewise C^1 boundary cure

$bQ = tr(Q)$ boundary of Q

$\overline{Q}$ = closed $\overline{\Delta}$

Pf: Case p outside $\overline{Q}$. Write $Q_0 = Q$.



• = midpoints of sides

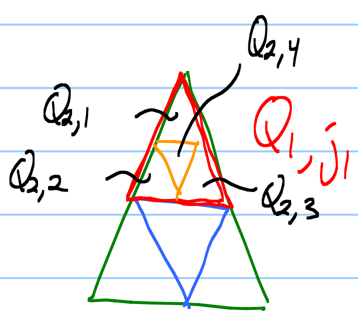
Aha! $I = \int_{Q_0} f \, dz = \sum_{j=1}^{4} \int_{Q_{1,j}} f \, dz$

Baby lemma: If $I = a_1 + a_2 + a_3 + a_4$, then $\exists j$ such that $|a_j| \geq \frac{|I|}{4}$.

Pf: $I = 0$, duh! $I \neq 0$: $|I| \leq \sum_1^4 |a_j|$.

If all $|a_j| < \frac{|I|}{4}$, ↯ $(|I| < |I|.$ ↯$)$

So, $\exists j_1$ such $\left| \int_{Q_{1,j_1}} f \, dz \right| \geq \frac{|I|}{4}$.



Let $Q_1 = Q_{1,j_1}$.

Repeat!

Get $\int_{Q_1} f \, dz = \sum_1^4 \int_{Q_{2,j}} f \, dz$

$\exists j_2$ such that

$$\left| \int_{\underset{\uparrow Q_2}{Q_{2,j_2}}} f \, dz \right| \geq \frac{1}{4} \underbrace{\left| \int_{Q_1} f \, dz \right|}_{\geq \frac{|I|}{4}} \geq \frac{|I|}{4^2}$$

Continue! Get

$$\overline{Q_0} \supset \overline{Q_1} \supset \overline{Q_2} \supset \cdots$$

$$\left| \int_{Q_n} f \, dz \right| \geq \frac{|I|}{4^n}$$

<u>Topology lemma</u> $K_n \neq \emptyset$ <u>compact</u> in $\mathbb{R}^2$,

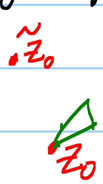
$$K_0 \supset K_1 \supset K_2 \supset \cdots$$

Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.

Furthermore, if $\operatorname{diam}(K_n) \longrightarrow 0$ as $n \to \infty$, then

$\bigcap_{n=0}^{\infty} K_n = \{z_0\}$, a single pt in K_0.

<u>Δ pf</u>: Take leftmost (or lower leftmost corner if "leftmost" is a vert line) of each Δ. x-coord is <u>nondecreasing</u>, bdd above. They <u>converge</u> to x_0. (completeness of $\mathbb{R}$). Similarly y-coord converge to y_0. $z_0 = x_0 + iy_0$ is the pt. No other z_0 because:



Q_n : $\operatorname{diam}(Q_n) < \operatorname{dist}(\tilde{z}_0, z_0)$

Back to Goursat pf: $\bigcap_{n=0}^{\infty} \overline{Q_n} = \{z_0\}$, $z_0 \in \overline{Q_0}$.

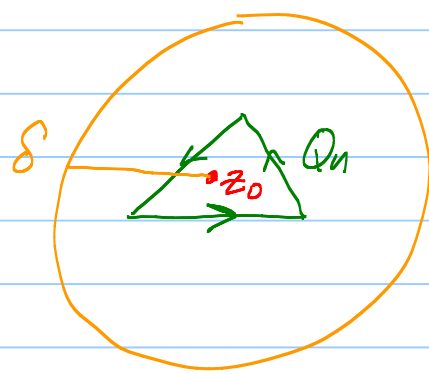
$$\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) + E(z)$$ ← defines $E(z)$, $z \neq z_0$. Define $E(z_0)=0$.

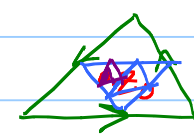
f complex diff'ble at $z_0 \neq p$. So $E(z) \to 0$ as $z \to z_0$.

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z-z_0)}_{Az+B} + \underbrace{E(z)(z-z_0)}_{\text{really small near } z_0}$$ ← true at $z=z_0$ too.

Note: $E(z)$ is continuous on Ω.

Let $\varepsilon > 0$. $\exists\, \delta > 0$ such that $|E(z)| < \varepsilon$ when $|z-z_0| < \delta$. $\exists\, N$ such that $\overline{Q_n} \subset D_\delta(z_0)$ if $n > N$.





Now $$\int_{Q_n} f\,dz = \underbrace{\int_{Q_n} Az+B\,dz}_{=0} + \int_{Q_n} E(z)(z-z_0)\,dz$$

So $$\left|\int_{Q_n} f\,dz\right| \le \underbrace{\left(\max_{z\in bQ_n} |E(z)(z-z_0)|\right)}_{\le\, \varepsilon \max_{z\in bQ_n}|z-z_0| \;\le\; \varepsilon\, \mathrm{Diam}(Q_n)} \mathrm{Length}(bQ_n)$$

Aha! $\mathrm{Diam}(Q_n) = \frac{1}{2^n}\mathrm{Diam}(Q_0)$

$$\mathrm{Length}(Q_n) = \frac{1}{2^n}\mathrm{Length}(bQ_0)$$

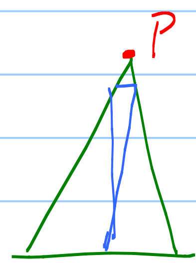
Hence, if $n > N$, then

$$\frac{|I|}{4^n} \le \left|\int_{Q_n} f\,dz\right| \le \varepsilon \frac{1}{4^n}\mathrm{Diam}(Q_0)\,\mathrm{Length}(bQ_0)$$

$|I| \le c\varepsilon$ where $c = \mathrm{Diam}(Q_0)\,\mathrm{Length}(bQ_0)$.

ε, arbitrary! Must be that $I = 0$. ✓

<u>Case p = vertex</u>:

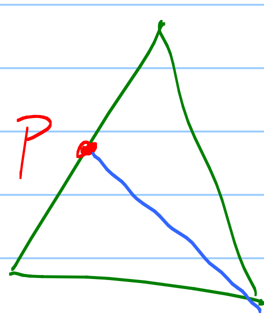


$$\int_Q = \text{Sum} = \int_{\text{tiny top } \Delta} f\,dz$$

f continuous at p, so bdd on $D_\delta(p)$. $|f| \le M$

$\left|\int_\Delta f\,dz\right| \le M\,\mathrm{Length}(b\Delta)$ can be made small.

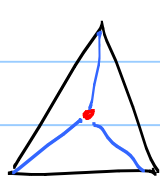
<u>Case $p \in$ side</u>:



$\int_Q f\,dz$ = Sum of 2

Reduce to p = vertex case

Case p inside



Reduce to p = vertex case