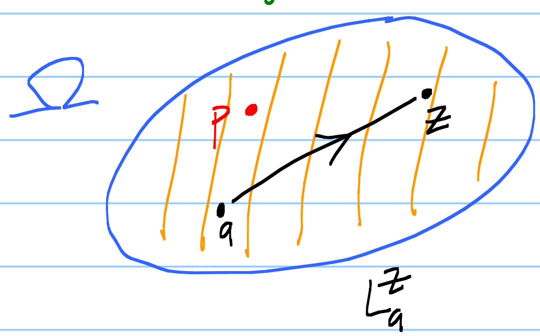
Lecture 7 Cauchy Theorem on a convex open set

Hwk 2 due in GS
Thurs, 11:59 pm

Thm: Suppose Ω is a convex open set in $\mathbb{C}$,
$f : \Omega \to \mathbb{C}$ is continuous on Ω,
f is analytic on $\Omega - \{p\}$. $(p \in \Omega)$

Then $\int_\gamma f\,dz = 0$ for any closed curve γ in Ω.



Pf: Plan: Given f, cook up F such that
$F' = f$ ← continuous!

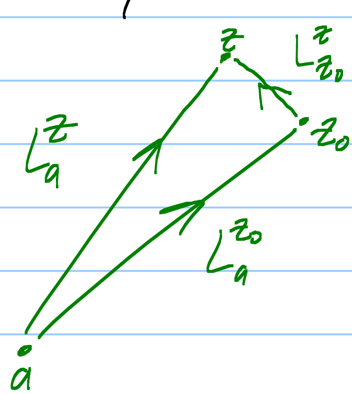
If we had such an F, then $\int_{L_a^z} f\,dw = \int_{L_a^z} F'\,dw = F(z) - \underbrace{F(a)}_{\text{a fixed, const.}}$

(↑ F. Thm Calc. #2)

Aha! $\frac{d}{dz}\Big(\underbrace{\int_{L_a^z} f\,dw}_{\text{Define to be } F(z)}\Big) = F'(z) = f(z)$

Why we're doing this: $\int_\gamma f\,dz = \int_\gamma F'\,dz = F(\text{END}) - F(\text{START})$
$= 0$ when γ closed.

Show $F' = f$: Key: Goursat!



$$\left(\int_{L_a^{z_0}} + \int_{L_{z_0}^z} + \underbrace{\int_{-L_a^z}}_{-\int_{L_a^z}}\right) f\,dw = 0$$

Aha! $$DQ = \frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_{L_{z_0}^{z}} f\ dw$$

Baby fact: $$\int_{L_{z_0}^{z}} c\ dw = \int_L \frac{d}{dw}[cw]\,dw = [cw]_{z_0}^{z} = c(z-z_0)$$

f continuous at z_0: $f(z) = f(z_0) + E(z)$

where $E(z)$ is cont. on Ω and $E(z) \to 0$ as $z \to z_0$.

$$DQ = \frac{1}{z-z_0}\left[\int_{L_{z_0}^{z}} f(z_0) + E(w)\ dw\right]$$

$$= \underbrace{\frac{1}{z-z_0}\cdot f(z_0)(z-z_0)}_{f(z_0)} + \underbrace{\frac{1}{z-z_0}\int_{L_{z_0}^{z}} E(w)\,dw}_{\mathcal{E}(z)}$$

where $$|\mathcal{E}(z)| \le \frac{1}{|z-z_0|}\underbrace{\left(\max_{w\in L_{z_0}^{z}} |E(w)|\right)}_{<\varepsilon} \underbrace{\text{Length}\,(L_{z_0}^{z})}_{|z-z_0|}$$

when $|z-z_0|<\delta$, some δ.

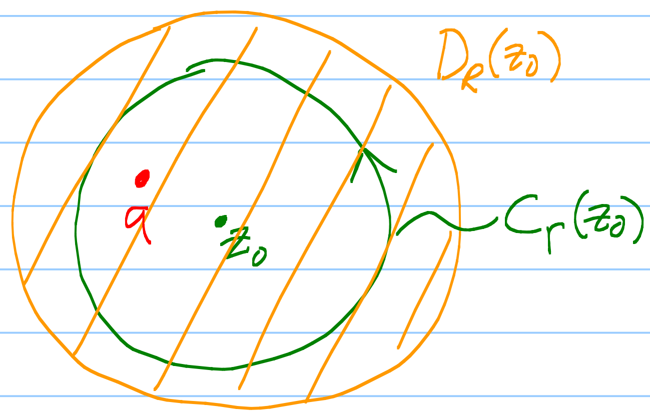
$\le \varepsilon$ when $|z-z_0|<\delta$. ✓

Cor: Analytic fcns on a convex open have an analytic antiderivative there given by $F(z) = \int_{L_a^z} f(w)\,dw$. ← a fixed in Ω

Future: Improve "convex open" to

$\underbrace{\text{simply connected}}_{\text{no holes}}$ $\underbrace{\text{domain}}_{\text{open, connected}}$.

Cauchy integral Formula on a disc



f analytic on $D_R(z_0)$

$C = C_r(z_0)$ = circle of radius r about z_0 in counterclockwise sense

$a \in D_r(z_0)$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Lemma:
$$\int_{C_r(z_0)} \frac{1}{z-a}\,dz = \begin{cases} 2\pi i & a \in D_r(z_0) \\ 0 & |z-z_0| > r \end{cases}$$

Cauchy on convex.

Remark: $f(z) \equiv 1$ in C.I.F.

Pf: $C = C_r(z_0)$: $z(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

$z'(t) = ire^{it}$

Easy case: $a = z_0$: $\displaystyle\int_C \frac{1}{z-z_0}\,dz$

$$= \int_0^{2\pi} \frac{1}{\underbrace{(z_0+re^{it})}_{z(t)} - z_0}\,\underbrace{ire^{it}}_{z'(t)}\,dt$$

$$= \int_0^{2\pi} i\,dt = 2\pi i \quad \checkmark$$

Define
$$H(w) = \int_C \frac{1}{z-w}\, dz$$

Claim:
$$H'(w) = \int_C \frac{d}{dw}\left[\frac{1}{z-w}\right] dz = \int_C \frac{1}{(z-w)^2}\, dz$$

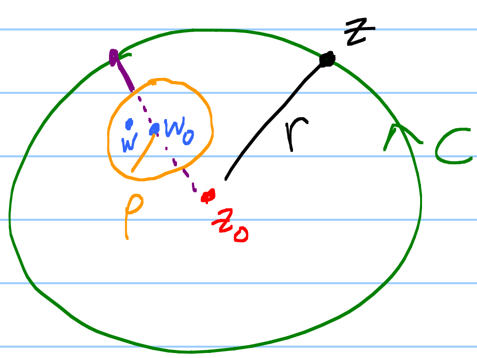
(HWK 2: prob 1)

Pf: $$DQ = \frac{H(w)-H(w_0)}{w-w_0} = \frac{1}{w-w_0}\int_C \underbrace{\frac{1}{z-w} - \frac{1}{z-w_0}}_{\frac{w-w_0}{(z-w)(z-w_0)}}\, dz$$

$$= \int_C \frac{1}{(z-w)(z-w_0)}\, dz \xrightarrow[\text{as } w \to w_0]{\text{guess}} \int_C \frac{1}{(z-w_0)^2}\, dw$$

$$DQ - \text{guess} = \int_C \underbrace{\frac{1}{(z-w)(z-w_0)} - \frac{1}{(z-w_0)^2}}_{\frac{w-w_0}{(z-w)(z-w_0)^2}}\, dz$$

$$|DQ - \text{guess}| = |w-w_0| \cdot \underbrace{\left|\int_C \frac{1}{(z-w)(z-w_0)^2}\, dz\right|}_{\text{just need to bound}}$$



$$\rho = \text{dist}(w_0, C)/2$$
$$= \left(r - |w_0 - z_0|\right)/2$$

$|z-w| \geq \rho \qquad |z-w_0| \geq 2\rho$

$$\left| \int_C \frac{1}{(z-w)(z-w_0)^2}\, dz \right| \leq \frac{1}{\rho\,(2\rho)^2} \cdot \underbrace{\text{Length}(C)}_{2\pi r}$$

Let $w \to w_0$ inside $D_\rho(w_0)$. ✓

Claim: $H'(w) \equiv 0$

$$H'(w) = \int_C \frac{d}{dw}\left[\frac{1}{z-w}\right] dz = \int_C \frac{1}{(z-w)^2}\, dz$$

$$\underset{\text{sneaky!}}{=} \int_C \frac{d}{dz}\left[\frac{-1}{z-w}\right] dz$$

$$= 0 \quad \text{by F. Thm Calc \#2 !}$$

So $H \equiv$ const. $H(z_0) = 2\pi i$. $c = 2\pi i$ ✓

Pf of Cauchy Int. Formula:

$$G(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases} \qquad a = P\,!$$

G cont. on $D_R(z_0)$, analytic on $D_R(z_0) - \{a\}$.

Cauchy on convex $\Rightarrow$ $\displaystyle\int_C G\, dz = 0$

$$\int_C \frac{f(z)-f(a)}{z-a}\,dz = 0$$

$$\int_C \frac{f(z)}{z-a} - f(a)\underbrace{\int_C \frac{1}{z-a}\,dz}_{2\pi i} = 0 \checkmark$$

Higher order Cauchy Integral Formulas

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \qquad \text{HWK 2 : Prob 1}$$

$$f''(a) = \frac{2}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Aha! This is how you see that

f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow$ ...

$\Rightarrow f'$ cont $\qquad$ f'' cont

Soon: Cauchy Int Formula $\Rightarrow$ f = power series !