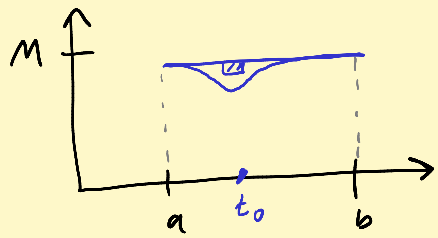
Lecture 11 The Maximum principle

Calc lemma $h:[a,b]\to\mathbb{R}$ continuous, $h(t)\le M$ for all t.

Then $\displaystyle\int_a^b h(t)\,dt = M(b-a) \implies h(t)\equiv M$ on $[a,b]$.



Key: Cauchy integral formula $\implies$ analytic, "the averaging property" on discs.

$C_r(z_0)$: $z(t)=z_0+re^{it}$, $0\le t\le 2\pi$. $z'(t)=ire^{it}$

$$f(z_0)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+re^{it})}{(z_0+re^{it})-z_0}\underbrace{ire^{it}\,dt}_{dz=z'(t)dt}$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt$$

$$=\frac{1}{2\pi r}\int_0^{2\pi} f(z_0+re^{it})\underbrace{r\,dt}_{ds}$$

$$=\text{Ave of } f \text{ on } C_r(z_0)$$

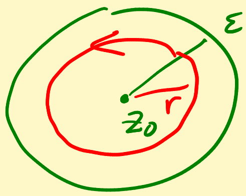
Fact: Ave prop $\implies$ Max principle

Maximum princ #1: Suppose f analytic on a domain Ω.

If $|f|$ attains a local max at $z_0\in\Omega$ (meaning that

$\exists\ \varepsilon > 0$ such that $D_\varepsilon(z_0) \subset \Omega$ and $|f(z)| \le |f(z_0)|$ for all $z \in D_\varepsilon(z_0)$), then f must be constant on Ω.

Pf: Suppose $|f|$ has a local max at z_0. Get $D_\varepsilon(z_0)$ as in statement.



$r < \varepsilon$

$$\underbrace{|f(z_0)|}_{M} = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + re^{it})\,dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0+re^{it})|}_{\le M}\,dt$$

$$\le \frac{1}{2\pi}\,2\pi\cdot M$$

See $\int_0^{2\pi} |f(z_0+re^{it})|\,dt = 2\pi M$, Calc lemma $\Rightarrow$

$$|f(z_0+re^{it})| \equiv M, \quad 0 \le t \le 2\pi.$$

$r < \varepsilon$ arbitrary. So $|f| \equiv M$ on $D_\varepsilon(z_0)$

Lemma $|f| \equiv M$ on $D_\varepsilon(z_0) \Rightarrow f \equiv$ const there.

Pf: $f(x+iy) = u(x,y) + i\,v(x,y)$. $\quad |f|^2 = M^2$

(*) $u^2 + v^2 = M^2$

Case $M=0$. $f \equiv 0$. ✓

Case $M \ne 0$.

$$\begin{cases} \frac{\partial}{\partial x}(*) : & 2u\,u_x + 2v\,v_x = 0 \\ \frac{\partial}{\partial y}(*) : & 2u\,\overset{=-v_x}{u_y} + 2v\,\overset{=u_x}{v_y} = 0 \end{cases}$$

Cancel 2's.

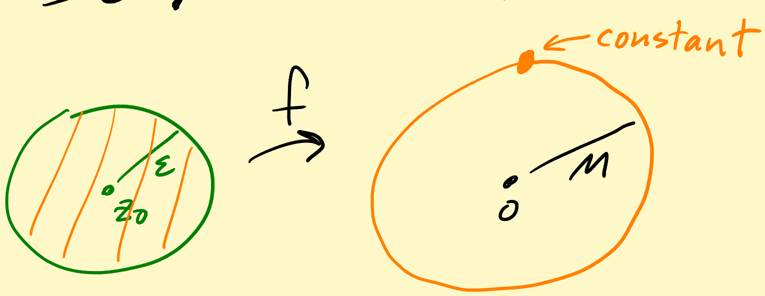
$$\underbrace{\begin{bmatrix} u_x & v_x \\ -v_x & u_x \end{bmatrix}}_{\text{det must be } = 0} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{\substack{\text{not zero vector} \\ u^2+v^2 \neq 0.}} = \vec{0}$$

$$\det = \underbrace{u_x^2 + v_x^2}_{|f'|^2}$$

Red box #1: $f' = u_x + i v_x$

So $f' \equiv 0. \implies f \equiv \text{const on } D_\varepsilon(z_0)$. ✓

Picture

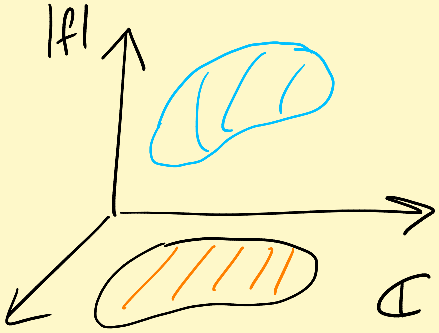


Pf of Max princ ending. $|f| \equiv M$ on $D_\varepsilon(z_0) \implies$

$f \equiv c$ on $D_\varepsilon(z_0)$. Identity thm $\implies$

$f \equiv c$ on Ω (because z_0 limit pt of

Z_{f-c} in Ω).



No place to sit on graph of $|f|$.

Max princ #2: Suppose Ω is a bonded domain ($\Omega \subset D_R(0)$ for some $R>0$), and f is continuous on the closure $\overline{\Omega}$, and f is analytic on Ω. Then $|f|$ attains its max modulus on $\overline{\Omega}$ at a pt in the boundary.

[$\exists$ z_0 in bndry where $|f(z_0)| = \underset{\overline{\Omega}}{\text{Max}}\, |f|$.]

Consequently: $\underset{\overline{\Omega}}{\text{Max}}\, |f| = \underset{\text{bndry}}{\text{Max}}\, |f|$

Pf.: f continuous $\Rightarrow$ $|f|$ continuous.

$\overline{\Omega}$ closed and bounded, and so, compact.

$|f|$ cont on compact attains it max.

Say $|f|$ is a max at $z_0 \in \overline{\Omega}$.

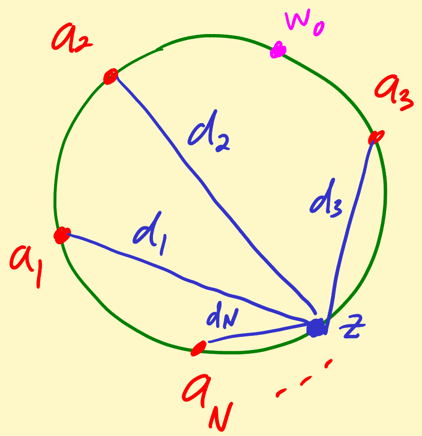
Case: z_0 in bndry. Done.

Case: $z_0 \in \Omega$. Max princ #1 $\Rightarrow f \equiv c$, a const.

Points in bndry are limit pts of a seq in Ω.

So continuity $\Rightarrow f = c$ on bndry too. ✓

Problem:



unit circle

Show that $\exists$ z with $|z|=1$ such $\underbrace{\prod_{n=1}^{N} d_n}_{|f(z)|} = 1$.

Let $f(z) = \prod_{n=1}^{N} (z - a_n)$

$$f(z) = z^N + \cdots + \prod_1^N a_n .$$

f(z) not constant

Aha! $|f(0)| = \left|\prod_1^N a_n\right| = 1$

f not const, Max princ $\Rightarrow \exists$ w_0 with $|w_0|=1$

such that $|f(w_0)| > 1$.

Aha! $f(a_n) = 0$. Intermediate Value Thm gives pt where $|f(e^{i\theta})| = 1$.

Remark: z^n on $D_1(0)$ shows no Minimum modulus thm, but if $f(z)$ non-vanishing on Ω, there is.

Pf: Apply Max princ to $1/f$.

Remark: Harmonic fcns satisfy ave prop too, and so max princ too.