Lecture 13 Complex logarithm fcns, isolated singularities of analytic fcns

Complex log fcn: $W = e^z = e^{x+iy} = \underset{\substack{\uparrow \\ r}}{e^x}\ \underset{\substack{\uparrow \\ e^{i\theta}}}{e^{iy}}$

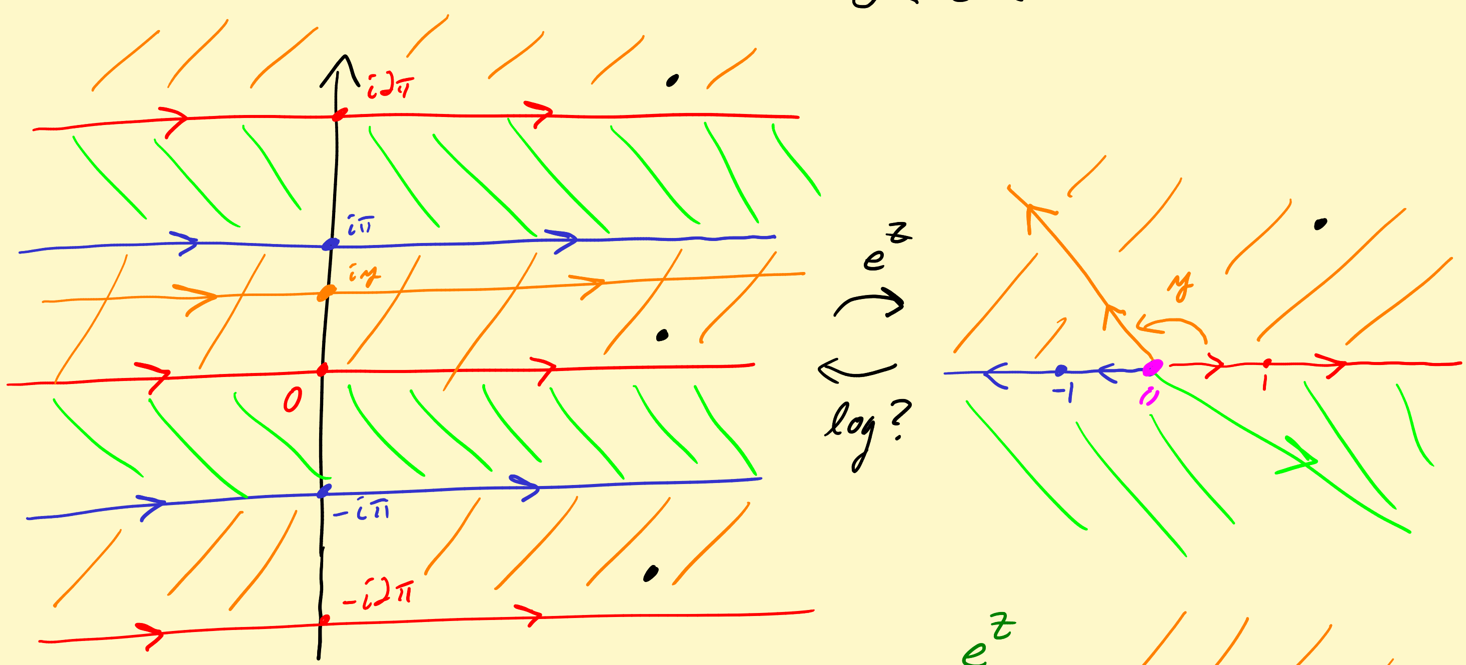
See $\underbrace{|w| = e^x}_{x = \mathrm{Ln}|w|}$, $y \in \arg w$

So $z = \mathrm{Ln}|w| + i\theta$ where $\theta \in \arg w$

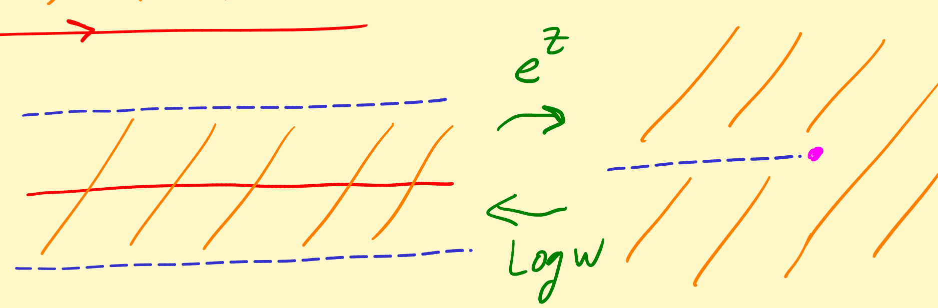
Def^n: $\underset{\substack{\uparrow \\ \text{Principal log}}}{\mathrm{Log}}\ w = \mathrm{Ln}|w| + i\,\underset{\substack{\uparrow \\ \text{Principle argument } \theta,\ -\pi < \theta \le \pi}}{\mathrm{Arg}}\ w$

$$\log w = \{\mathrm{Ln}|w| + i\theta : \theta \in \arg w\}$$

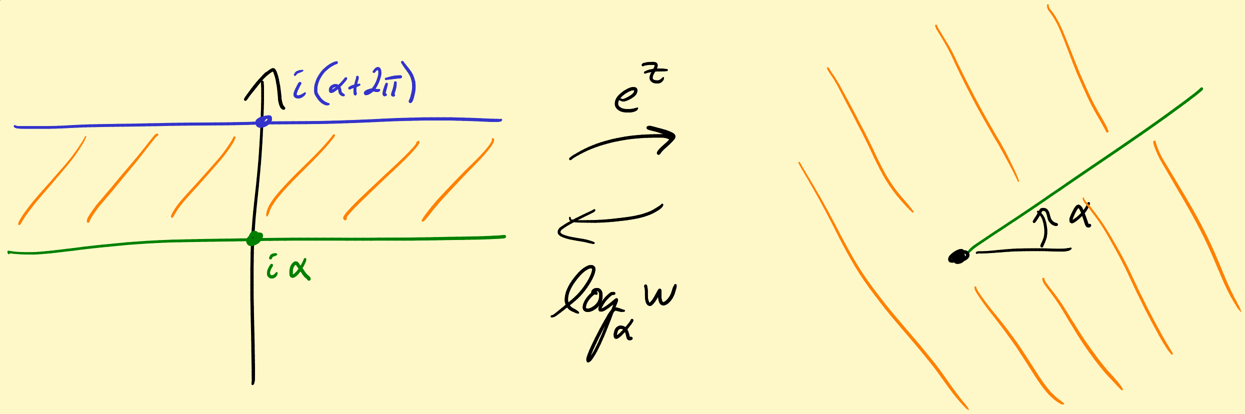
Graph of e^z $e^{x+iy} = e^x e^{iy}$ $-\infty < x < \infty$, $0 < e^x < \infty$



Principal Log:

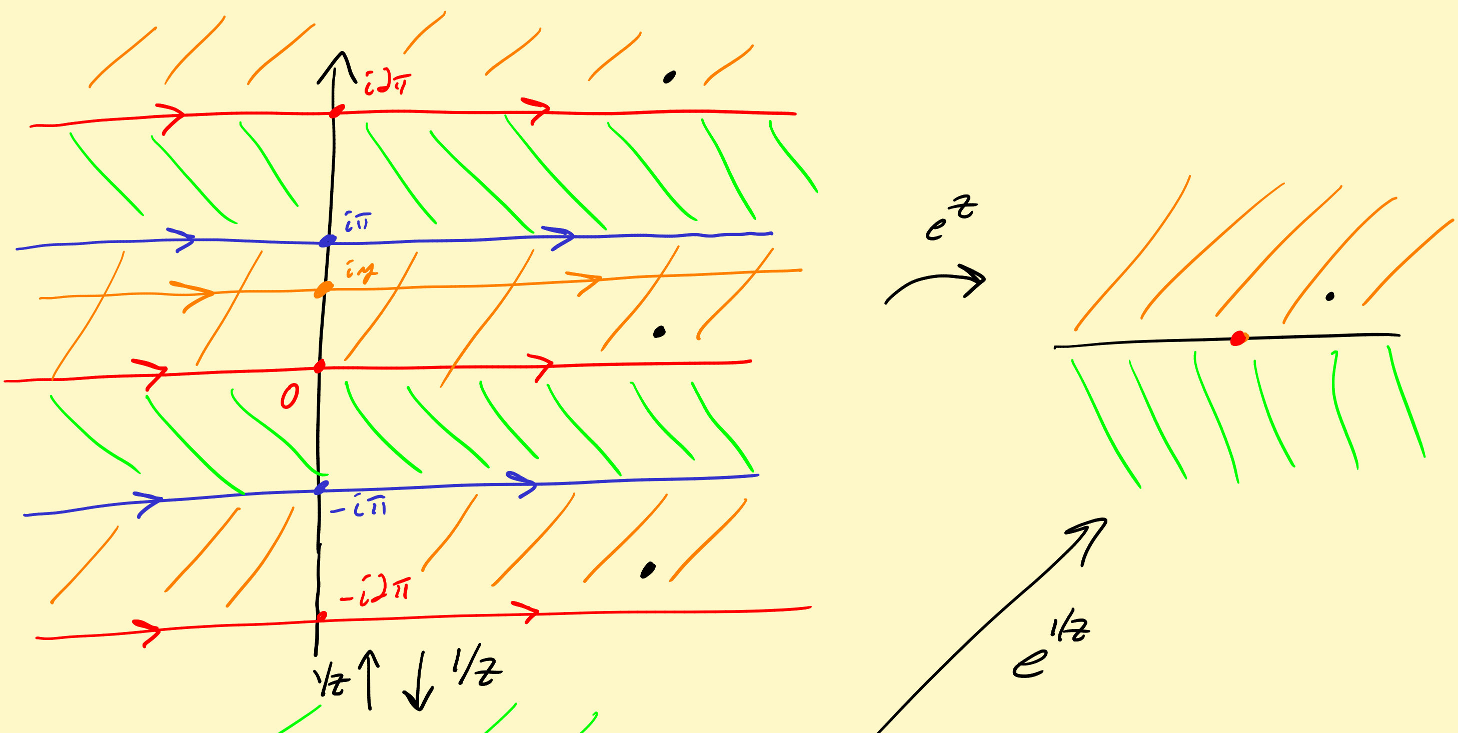


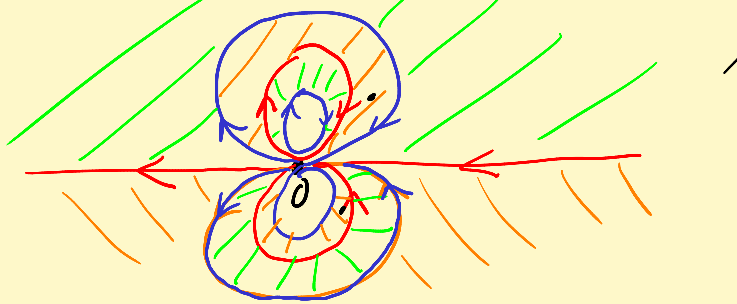
$\mathrm{Log}\, w : \mathbb{C} - (-\infty, 0] \xrightarrow[\text{analytic}]{1\text{-}1,\ \text{onto}} \{z : -\pi < \mathrm{Im}\, z < \pi\}$



$\log_\alpha w : \{\mathbb{C} - \{re^{i\alpha}\} : 0 < r < \infty\} \xrightarrow[\text{onto}]{1\text{-}1}$ horiz strip
analytic

$e^{1/z}$ has a mind blowing singularity at $z = 0$!



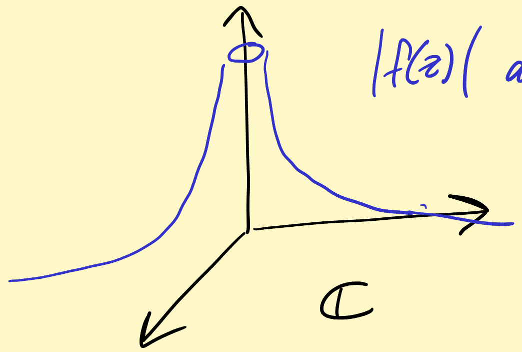


Pick small $D_\varepsilon(0) - \{0\}$.
$e^{1/z}$ maps $D_\varepsilon(0) - \{0\}$
onto $\mathbb{C} - \{0\}$ infinite-to-one!

0 is called an "essential singularity" of $e^{1/z}$.

Other types of behavior 1) $\frac{1}{z^3}$ blows up at $z=0$, meaning $\left|\frac{1}{z^3}\right| = \frac{1}{|z|^3} \longrightarrow \infty$ as $z \to 0$.

$\frac{1}{z^3}$ has a "pole" at $z=0$.



2) $$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = \frac{\cancel{z}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}{\cancel{z}}$$

power series converges for same z that $\sin z$ series does. $R = \infty$.

$z=0$ is a "removable singularity"

Remove it:

$$h(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

Big fact: These three behaviors are the only possible behaviors at an "isolated singularity" (meaning $\exists\, \varepsilon > 0$ such that fcn is analytic on $D_\varepsilon(z_0) - \{z_0\}$).

Def^n: $\lim_{z \to a} f(z) = \infty$ means: Given $M > 0$, there is a $\delta > 0$ such that $|f(z)| > M$ when $|z-a| < \delta$

and $z \neq a$. [f has an isolated sing at a, $D_\rho(a) - \{a\}$. $0 < \delta < \rho$.]

f has a pole at a.

Def^n: f has an essential sing if, for every ε with $0 < \varepsilon < \rho$, $f(D_\varepsilon(a) - \{a\})$ is dense in $\mathbb{C}$ (meaning every pt in $\mathbb{C}$ is a limit point of $\{w : w = f(z)$ for some $z \in D_\varepsilon(a) - \{a\}\}$.

Goursat/Morera removable singularity thm. Suppose f is continuous on Ω and analytic on $\Omega - \{a\}$, $a \in \Omega$. Then a is a removable sing for f.

Pf: Goursat $\Rightarrow \int_\Delta f\, dz = 0$ for all Δ's in Ω.

Morera $\Rightarrow$ f analytic on Ω. ✓

Riemann removable singularity thm: Suppose f analytic on $D_\rho(a) - \{a\}$ and bounded there (meaning $\exists M > 0$ such that $|f(z)| < M$, z in set). Then a is a removable sing for f.

Way false $\mathbb{R} \to \mathbb{R}$: $\sin \frac{1}{x}$

Pf: Define $$G(z) = \begin{cases} f(z)(z-a) & z \neq a \\ 0 & z = a \end{cases}$$

<u>Note</u>: $\lim_{z \to a} G(z) = 0$. So G cont at $z=a$.

Goursat/Morera thm $\Rightarrow$ G analytic on $D_\rho(a)$.

<u>and</u> $G(a)=0 \Rightarrow G(z) = \underbrace{G(a)}_{0} + G'(a)(z-a) + \cdots$

$$= (z-a)\underbrace{\left[G'(a) + \frac{G''(a)}{2!}(z-a) + \cdots \right]}_{g(z) \text{ analytic}}$$

Aha! $\underbrace{\frac{G(z)}{z-a}}_{g(z)} = f(z)$, $z \neq a$

Setting $f(a) = g(a) = G'(a)$ makes f analytic on $D_\rho(a)$.

<u>Remark</u>: Books $G(z) = \begin{cases} f(z)(z-a)^2 & z \neq a \\ 0 & z = a \end{cases}$

See G analytic on $D_\rho(a)$. $G(a) = 0$
$G'(a) = 0$