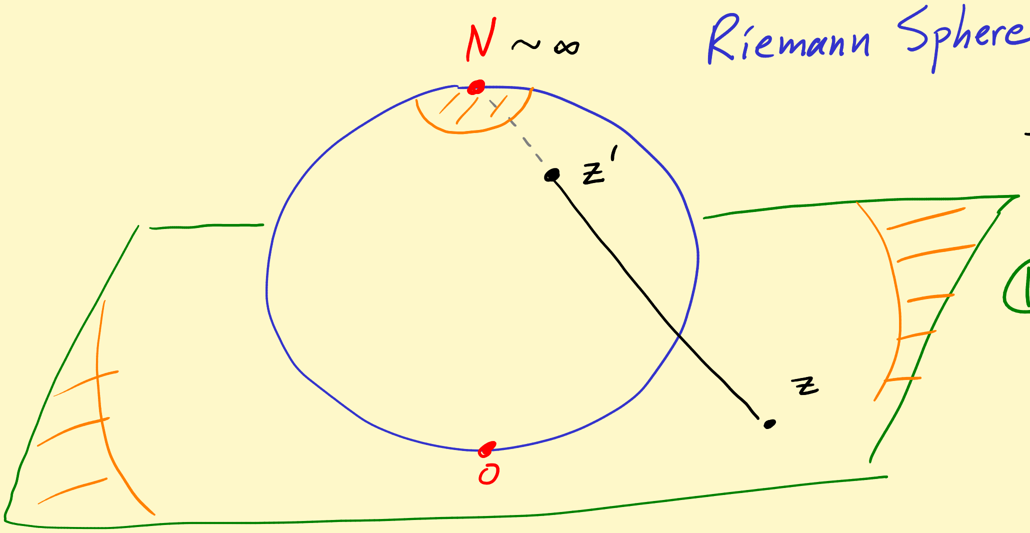
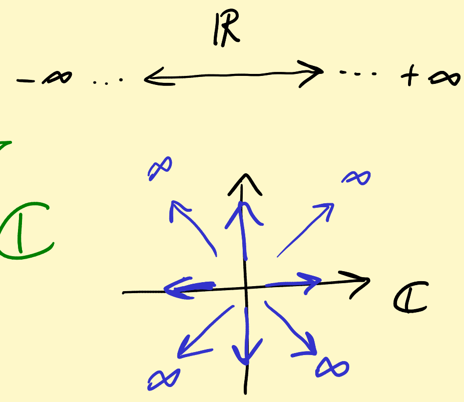
Lecture 14 Isolated singularities, point at ∞, Picard theorems

Riemann Sphere

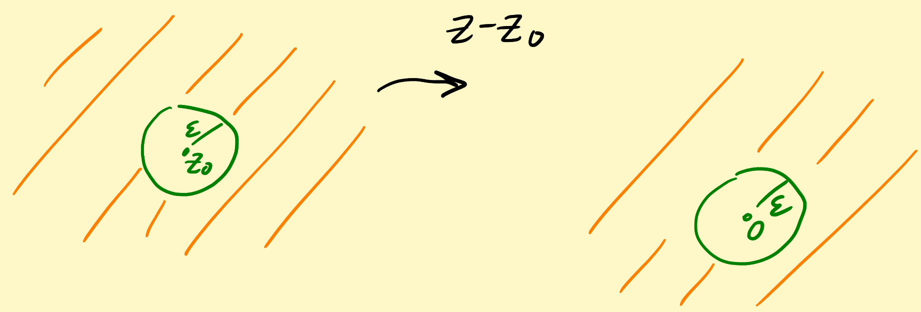


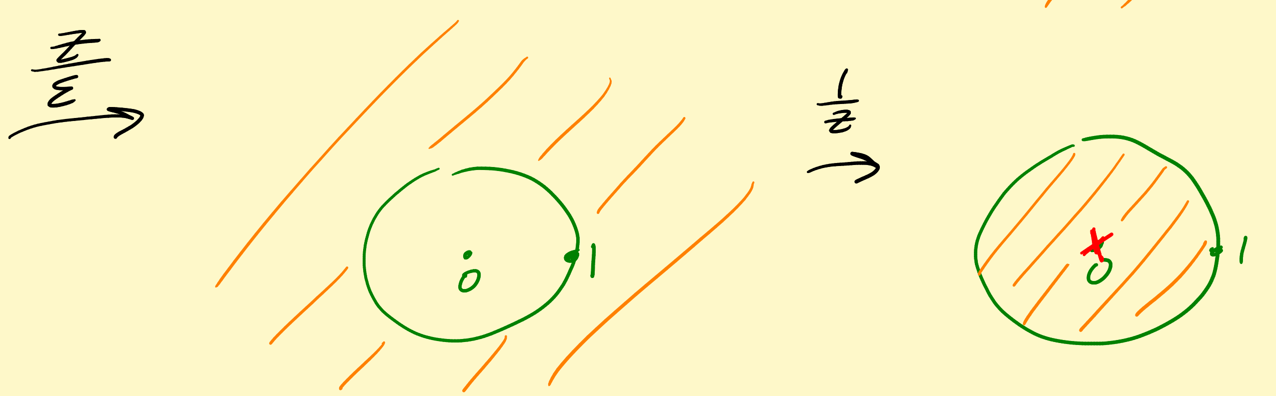


$$D_\varepsilon(\infty) = \left\{ z : |z| > \frac{1}{\varepsilon} \right\} \cup \{\infty\}$$

$\lim\limits_{z \to z_0} f(z) = \infty$: Setting $f(z_0) = \infty$ makes f continuous on Riemann sphere.

Important map





$$h(z) = \frac{\varepsilon}{z - z_0} : \quad \mathbb{C} - \overline{D_\varepsilon(z_0)} \xrightarrow[\text{onto}]{1-1} D_1(0) - \{0\}.$$

Casorati-Weierstraß Thm: If an entire fcn misses an open set, it must be constant.

Pf: Suppose f entire and misses $D_\varepsilon(z_0)$.

Aha! Get bdd entire fcn! Liouville's $\Longrightarrow \frac{\varepsilon}{f(z)-z_0} \equiv c \neq 0$

$\Longrightarrow f$ const.

Thm: Only 3 kinds of isolated sing: Removable, Pole, Essential.

Pf: Suppose f analytic on $D_\rho(z_0) - \{z_0\}$.

Suppose z_0 is not an essential singularity.

Then $\exists\ \varepsilon$ with $0 < \varepsilon < \rho$ such that

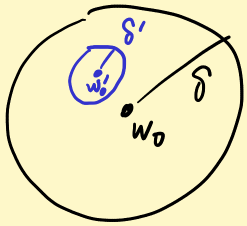
$\mathcal{A} = f\left(D_\varepsilon(z_0) - \{z_0\}\right)$ is not dense in $\mathbb{C}$.

Then $\exists\ w_0 \in \mathbb{C}$ that is not a limit of $\mathcal{A}$.

Case $w_0 \notin \mathcal{A}$: Then $\exists\ \delta > 0$ such $D_\delta(w_0) \cap \mathcal{A} = \emptyset$

f maps to outside of this disc

Case $w_0 \in \mathcal{L}$: Then $\exists\ \delta > 0$ with $D_\delta(w_0) \cap \mathcal{L} = \{w_0\}$.



Aha! Replace w_0 by w_0', δ by δ'

f maps $D_\varepsilon(z_0) - \{z_0\}$ to outside of $D_\delta(w_0)$. Compose with $h(z) = \frac{\delta}{z - w_0}$ to get a bdd fcn

$\frac{\delta}{f(z) - w_0}$ on $D_\varepsilon(z_0) - \{z_0\}$. Now Riemann removable sing thm (R.R.S.T.) $\Rightarrow$

$F(z) = \frac{\delta}{f(z) - w_0}$ can be given a value

at z_0 to make it analytic on $D_\varepsilon(z_0)$.

Solve for $f(z) = w_0 + \frac{\delta}{F(z)}$

Case $F(z_0) \neq 0$: See z_0 is removable.

Case $F(z_0) = 0$: z_0 is a pole.

Zeroes and poles Zeroes: $f(z) = a_N (z-a)^N + \cdots$

$$= (z-a)^N \underbrace{\left[a_N + a_{N+1}(z-a) + \cdots \right]}_{F(z)}$$

$$a_N = \frac{f^{(N)}(a)}{N!} \neq 0$$

f has a <u>zero</u> of order N at a $\iff$

$$f(z) = (z-a)^N F(z)$$ where F is analytic on some disc about a and $F(a) \neq 0$.

<u>Poles:</u> f has a <u>pole</u> of order N at a $\iff$

$$f(z) = (z-a)^{-N} F(z)$$ (same thing about F)

<u>why:</u> If a is a pole of f, then $\lim_{z \to a} f(z) = \infty$.

Take $M=1$. Get $\delta > 0$ such that $|f(z)| > 1$ when $z \in D_\delta(a) - \{a\}$. So $f(z)$ is nonvanishing there and $\left|\frac{1}{f(z)}\right| < 1$ on $D_\delta(a) - \{a\}$.

RRST $\Rightarrow$ $F(z) = \frac{1}{f(z)}$ can be given a value at a to make it analytic on $D_\delta(a)$. That value is $\underline{\underline{0}}$ because $|f(z)| \to \infty \Rightarrow \frac{1}{f(z)} \to 0$ as $z \to a$.

a is a clearly an <u>isolated</u> zero (because $\left|\frac{1}{f}\right| > 1$ on punctured disc). So it has an order N.

Get $\frac{1}{f(z)} = (z-a)^N G(z)$. $G(a) \neq 0$

$$f(z) = (z-a)^{-N} \underbrace{\frac{1}{G(z)}}_{F(z)}$$ ✓

Routine for poles: $f(z) = (z-a)^{-N} F(z)$

$$= \frac{1}{(z-a)^N}\left[A_0 + A_1(z-a) + A_2(z-a)^2 + \cdots\right]$$

$$= \underbrace{\frac{A_0}{(z-a)^N} + \frac{A_1}{(z-a)^{N-1}} + \cdots + \frac{A_{N-1}}{z-a}}_{\substack{\text{Principal part of } f \text{ at pole } a. \\ R(z)}} + \left(\begin{array}{c}\text{Convergent power series} \\ \text{same R of C as } F\end{array}\right)$$

Fact: $R(z)$ is uniquely determined by the property that $f(z) - R(z)$ has a removable sing at a.

Why: If R_1 and R_2 both do this, the $R_1 - R_2$ has a removable sing at a. Show $N_1 = N_2$ and each coeff must be same.

Picard's little theorem: Suppose f is entire. Then f is either a polynomial, or f takes every value in $\mathbb{C}$ infinitely many times with one possible exception (like e^z misses zero).

In particular, an entire fcn that misses two values must be constant.

Remark: e^z misses 0. $\cos z$ misses nothing.

<u>Picard's Big theorem</u>: If f analytic on $D_\rho(z_0)-\{z_0\}$ and has an essential sing at z_0, f maps $D_\varepsilon(z_0)-\{z_0\}$ ∞-to-one on $\mathbb{C}$ with one possible exception (like $e^{1/z}$ at $z=0$) $(0<\varepsilon<\rho)$