Lecture 16 Schwarz lemma, log and roots, Open mapping theorem

Schwarz lemma Suppose f is analytic on $D_1(0)$ and

$f: D_1(0) \longrightarrow D_1(0)$ with $f(0)=0$.

Then A) $|f(z)| \le |z|$ for $z \in D_1(0)$

B) $|f'(0)| \le 1$

and if equality occurs in (A) for some $z \neq 0$,

or if equality occurs in (B), then

$f(z) = \underbrace{\lambda z}_{\substack{\text{rotation} \\ \text{by } \alpha.}}$ for some unimodular const $\lambda = e^{i\theta}$.

Pf: $\dfrac{f(z)}{z} = \dfrac{f(z)-f(0)}{z-0}$

Let $F(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$

F has a removable sing at $z=0$. It's analytic on $D_1(0)$.

Let $z_0 \in D_1(0)$. Pick r with $|z_0| < r < 1$.

Max princ: $|F(z_0)| \le \max_{\overline{D_r(0)}} |F| = \max_{|z|=r} |F(z)|$

$= \max_{|z|=r} \dfrac{|f(z)|}{r} < \dfrac{1}{r}$

Aha! Let $r \nearrow 1$. See $|F(z_0)| \le 1$.

$\Longrightarrow$ (A), $z \neq 0$. $z=0$ case true too. (A) ✓

and $\Longrightarrow$ (B)

<u>Part 2</u>: If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then

$|F(z_0)| = 1$. Max princ $\Longrightarrow$ $F(z) =$ const λ

of modulus 1. So $\frac{f(z)}{z} = \lambda$, $z \neq 0$.

So $f(z) = \lambda z$ for $z \neq 0$. Holds for $z=0$ too. ✓

If $|f'(0)| = 1$, then $|F(0)| = 1$ and we get same conclusion.

Log and roots on a convex open set (or any set where Cauchy thm holds)

<u>Think</u>: Given <u>nonvanishing</u> analytic fcn $F(z)$.

If G complex log fcn for F: $F(z) = e^{G(z)}$.

Hmmm. $F'(z) = e^{G(z)} \, G'(z)$

$$\text{Aha!} \quad \frac{F'(z)}{F(z)} = G'(z)$$

Candidate for $G(z)$ is a complex anti-deriv of $\frac{F'}{F}$

<u>Pf</u>: On a convex open set:

$$G(z) \underset{\text{def}}{=} \int_{L_a^z} \frac{F'(w)}{F(w)}\,dw \quad \text{is an anti-deriv.}$$

So $G'(z) = \dfrac{F'(z)}{F(z)}$. Is $e^G = F$?

$G' = \dfrac{F'}{F}$

Almost! $$\frac{d}{dz}\left(\frac{F}{e^G}\right) = \frac{F'e^G - F\cdot\left[e^G G'\right]}{(e^G)^2}$$

$$\equiv 0$$

So $\dfrac{F}{e^G} = c$, a const. $c \neq 0$.

$F = c\,e^G$. Hmmm. Pick $b \in \log c$.

$= e^b e^G = e^{(\underline{\underline{b+G}})}$ this is the fcn we want. ✓

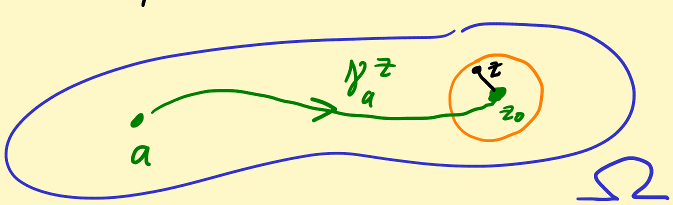
Ω a domain where Cauchy thm holds $\left(\int_\gamma f\,dz = 0,\ \gamma \text{ closed}\right)$.

Define $$G(z) = \int_{\gamma_a^z} \frac{F'(w)}{F(w)}\,dw$$ γ_a^z any curve in Ω starting at fixed a, going to z.

Step 1: G is well defind.

Cauchy ✓ $\quad \gamma_a^z \cup \left(-\tilde{\gamma}_a^z\right)$ closed curve

Step 2 :



$\gamma_a^z = \gamma_a^{z_0} \cup L_{z_0}^z$

Step 3: $G' \equiv \frac{F'}{F}$ on Ω.

$$\frac{d}{dz}\left(\frac{F}{e^G}\right) \equiv 0.$$

Need Ω domain to conclude $\frac{F}{e^G} \equiv c$.

Open mapping theorem Non-constant analytic fcns map open sets to open sets. (OMT)

Way false $\mathbb{R} \to \mathbb{R}$: $f(t) = 1 - t^2$

$$f(\underbrace{(-1,1)}_{\text{open}}) = \underbrace{(0,1]}_{\text{not open}}$$

Suppose f analytic on a domain Ω.

$$f(U) = \{f(z) : z \in U\}$$

$$f^{-1}(U) = \{z \in \Omega : f(z) \in U\}$$

Fact: f continuous $\iff$ $f^{-1}(U)$ open when U is open.

Cor of OMT: $f : \Omega_1 \xrightarrow[\text{onto}]{1-1} \Omega_2$ analytic.

1) Then $F = f^{-1}$ is continuous on Ω_2,

2) $(1) \Rightarrow F$ is analytic!

Pf: (1): Fact + OMT ✓

(2): $w = f(z)$ $\qquad z = F(w)$

$w_0 = f(z_0)$ $\qquad z_0 = F(w_0)$

F cont $\Rightarrow$ $\boxed{\underbrace{F(w)}_{z} \to \underbrace{F(w_0)}_{z_0} \text{ as } w \to w_0}$

$$DQ = \frac{F(w)-F(w_0)}{w-w_0} = \frac{z-z_0}{\underbrace{f(z)-f(z_0)}_{\text{not zero}}} \longrightarrow \frac{1}{f'(z_0)} \quad \text{as } w \to w_0.$$

So $F'(w_0)$ exists and Inverse fcn thm $F'(w_0) = \dfrac{1}{f'(z_0)}$

$= \dfrac{1}{f'(F(w_0))}$ holds.

$\mathbb{R} \to \mathbb{R}$ example: $h(t) = t^3$ $\quad h: (-1,1) \xrightarrow[\text{onto}]{1\text{-}1} (-1,1)$

h^{-1} not diff'ble on $(-1,1)$.

Argument principal for a disc: Suppose f analytic on $D_R(z_0)$ and $0<r<R$ and f has no zeroes on $C_r(z_0) = \{z : |z-z_0| = r\}$. Then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\,dz = \left(\begin{array}{l}\#\text{zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with multiplicity}\end{array}\right)$$

Key: $f(a)=0$. $\quad f(z) = (z-a)^N F(z)$ near a, $F(a) \neq 0$.

$$\frac{f'(z)}{f(z)} = \frac{N(z-a)^{N-1}F(z) + (z-a)^N F'(z)}{(z-a)^N F(z)}$$

$$= \underbrace{\frac{N}{z-a}} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{analytic near } a}$$

Aha! $\frac{f'}{f}$ has a simple pole at a!

$$N = \mathrm{Res}_a \frac{f'}{f} .$$