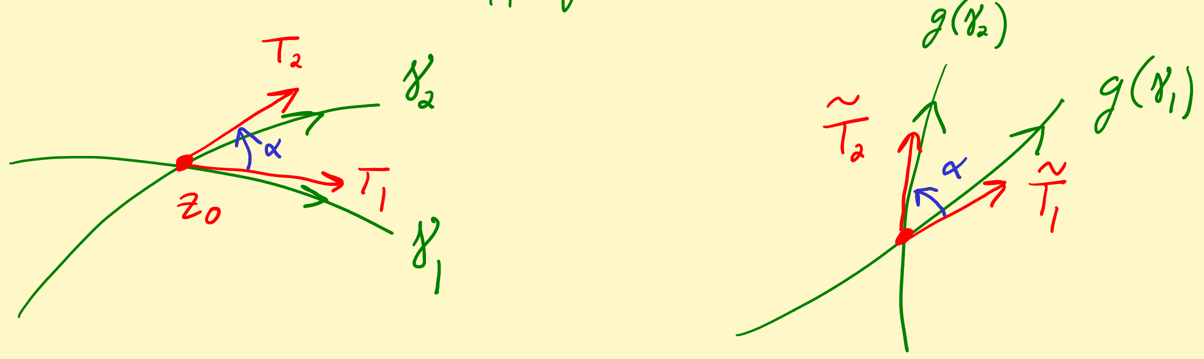
Lecture 19 Conformal mapping, zeroes of harmonic fcns, MA 525

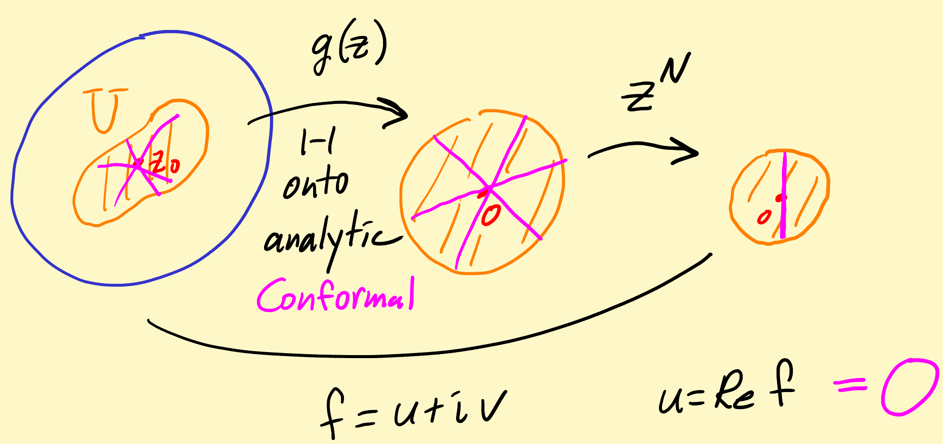


If g analytic and 1-1, then $g'(z_0) \neq 0$ and g is conformal at z_0: Angles and their sense are preserved.

This fact plus the Local Mapping Theorem allow us to understand zero sets of harmonic fcns:

Suppose u is a C^2-smooth real valued harmonic, $u(z_0)=0$, $u \not\equiv 0$. Step 1: Let v be a harm conj for u. Can assume $v(z_0)=0$ too. Let $f=u+iv$.

$f(z_0)=0$ and $f \not\equiv 0$. So f has a zero of some order N. LMT yields



$(w_0 = 0)$

See that zero sets of real harmonic fcns are a union of smooth curves. When they cross, they cross as in picture: angles $= \frac{2\pi}{N}$. $N \in \mathbb{N}$.

Converse to $(g \text{ analytic}, g' \neq 0) \Longrightarrow g$ conformal is "almost" true. $f(x+iy) = u(x,y) + i v(x,y)$

Suppose u, v $\mathcal{C}^1$-smooth and $\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \neq 0$.

If f is conformal, then f analytic.

Pf: $J = \underbrace{\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}}_{\substack{\text{linear map} \\ \text{causes rotation} \\ \text{by } \theta}} \underset{\substack{\uparrow \\ \text{analytic} \\ \text{geometry}}}{=} \begin{bmatrix} R\cos\theta & -R\sin\theta \\ R\sin\theta & R\cos\theta \end{bmatrix}$

See C-R eqn!

MA 525 $\quad f = u + iv$

525 Def^n: f analytic when u, v $\mathcal{C}^1$-smooth and satisfy C-R eqns.

Inverse fcn thm: $J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \underbrace{\begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}}_{\text{C-R eqns}} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$

$$\det J = (u_x)^2 + (v_x)^2 = \Big|\underbrace{u_x + i v_x}_{f'}\Big|^2$$

red box #1

So $f' \neq 0 \Rightarrow \det J \neq 0$.

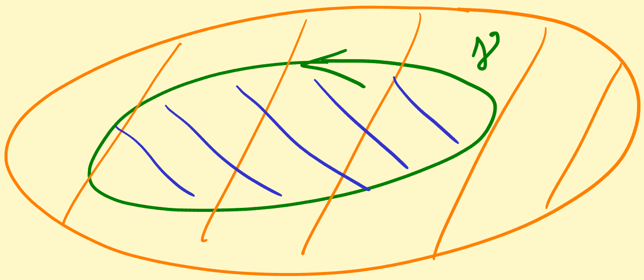
$\mathbb{R}^2 \to \mathbb{R}^2$ I.F.T applies. Get $f^{-1} = U + iV$

is $\mathcal{C}^1$-smooth too and

$$\left[\begin{matrix} a & b \\ c & d \end{matrix}\right]^{-1} = \frac{1}{\det}\left[\begin{matrix} d & -b \\ -c & a \end{matrix}\right]$$

$$J_{f^{-1}} = J_f^{-1} = \underbrace{\frac{1}{A^2+B^2}\left[\begin{matrix} A & -B \\ B & A \end{matrix}\right]}_{\text{Aha! See C-R eqns so } f^{-1} \text{ is analytic.}}$$

525 Cauchy Thm



f analytic on bigger open set

Ω = inside of γ

Then $\int_\gamma f \, dz = 0$ $\qquad \gamma: z(t), \; a \le t \le b$

$$\int_a^b \Big[\underbrace{u(x(t),y(t)) + i\,v(x(t),y(t))}_{f(z(t))}\Big] \Big(\underbrace{\dot{x}(t) + i\dot{y}(t)}_{z'(t)}\Big)\,dt$$

$$= \int_a^b (u\dot{x} - v\dot{y})\,dt + i\int_a^b (u\dot{y} + v\dot{x})\,dt$$

$$= \left(\int_\gamma u\,dx - v\,dy\right) + i\left(\int_\gamma u\,dy + v\,dx\right)$$

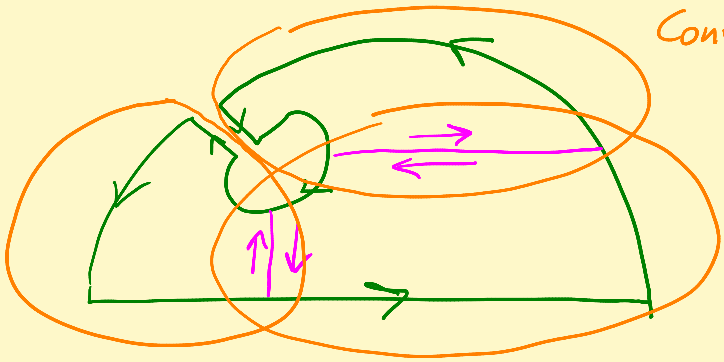
$$= \left(\iint_\Omega \underbrace{-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{CR\#2 \,\equiv 0} \,dA\right) + i\left(\iint_\Omega \underbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}_{C\text{-}R\#1 \,\equiv 0} \,dA\right) = 0$$

Green's thm

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Baby Residue Theorem on Toy regions

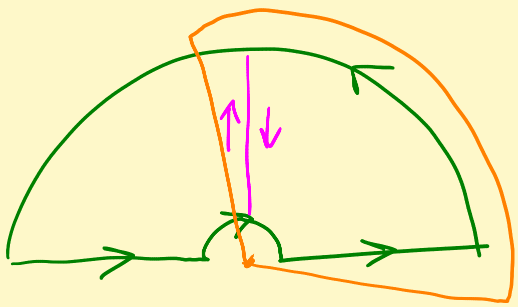
Toy region



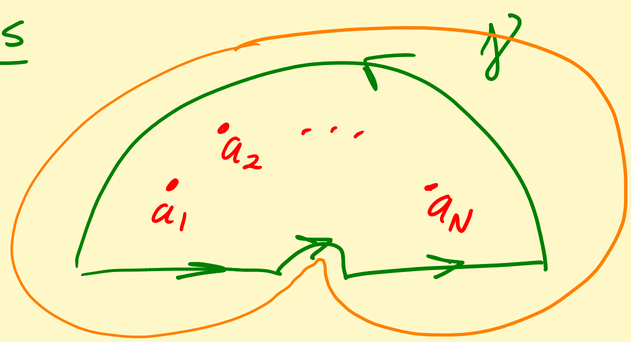
Can cut up to see Cauchy on convex $\Rightarrow \int_\gamma f\,dz = 0$

when f analytic on bigger open set.

EX:



Baby Residue thm on Toys



f analytic on bigger open set with finitely many poles $\{a_j : j=1,\ldots,N\}$ inside γ

Then $$\int_\gamma f\,dz = 2\pi i \sum_{n=1}^{N} \operatorname{Res}_{a_n} f$$

Pf: Let $R_n(z) = \dfrac{A_N}{(z-a_n)^N} + \cdots + \dfrac{A_1}{(z-a_n)}$ $\mathrm{Res}_{a_n} f$

be the principal part of f at a_n.

Now $f - \sum_{n=1}^{N} R_n(z)$ has removable sing at each a_n.

Cauchy: $\int_\gamma f\,dz - \int_\gamma \sum_{n=1}^{N} R_n(z)\,dz = 0$

Aha! $\dfrac{1}{(z-a)^k} = \dfrac{d}{dz}\left[\dfrac{1}{-k+1}(z-a)^{-k+1}\right]$ if $k \neq 1$.

$$\int_\gamma R_n(z)\,dz = 0 + 0 + 0 + \cdots + 0 + \underbrace{\int_\gamma \frac{A_1}{z-a_n}\,dz}_{2\pi i A_1}$$

Next time.