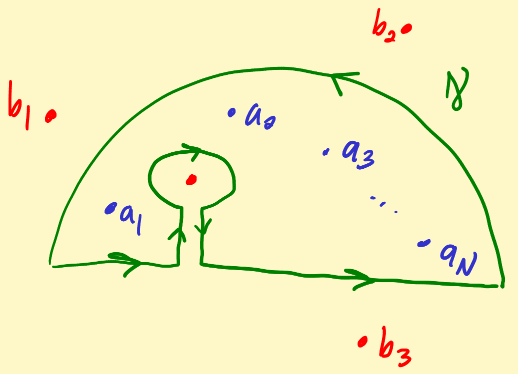
Lecture 20 Fun with the Baby residue theorem on Toy regions

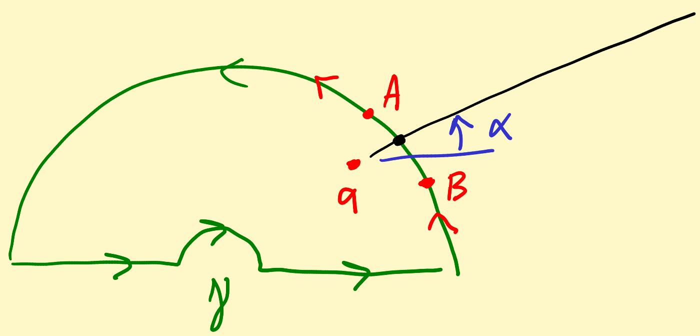


$$\int_\gamma f\,dz = 2\pi i \sum_{n=1}^{N} \text{Res}_{a_n} f$$

Last part of proof:
$$\underbrace{\int_\gamma \frac{1}{z-a}\,dz}_{H(a)} = \begin{cases} 2\pi i & a \text{ inside } \gamma \\ 0 & a \text{ outside } \gamma \end{cases}$$

Recall: $H'(a) = \int_\gamma \frac{1}{(z-a)^2}\,dz = \int_\gamma \frac{d}{dz}\left[\frac{-1}{z-a}\right] dz \equiv 0$

So $H(a) \equiv$ const inside γ.



Choose a near boundary so the α- branch cuts the boundary spot at an angle

$$I = \int_\gamma \frac{1}{z-a}\,dz = \lim_{\substack{A \text{ slides down} \\ B \text{ slides up}}} \int_{\gamma_A^B} \underbrace{\frac{1}{z-a}}_{\frac{d}{dz}\left[\log_\alpha(z-a)\right]}\,dz$$

$$\log_\alpha(z-a) = \text{Ln}|z-a| + i\theta, \quad \theta \in \arg(z-a), \ \alpha < \theta < \alpha + 2\pi$$

$$I = \lim \left[\left(\text{Ln}|B-a| + i\,(\alpha + 2\pi - \varepsilon_1)\right) - \left(\text{Ln}|A-a| + i\,(\alpha + \varepsilon_2)\right)\right]$$

$$= 2\pi i$$

Fact: $\frac{d}{dz} \log_\alpha z = \frac{1}{z}$

Pf: $e^{\log_\alpha z} = z$

So $$\underbrace{\frac{d}{dz}\left(e^{\log_\alpha z}\right)}_{\underbrace{e^{\log_\alpha z}}_{=z}\frac{d}{dz}\log_\alpha z} = \frac{d}{dz}(z) = 1$$

Thm: Suppose P, Q polynomials and

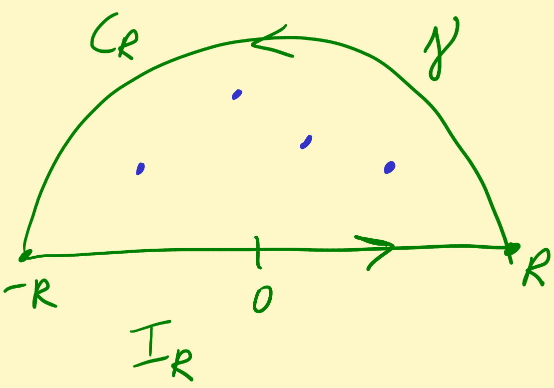
1) $\deg Q \geq \deg P + 2$

2) Q has no zeroes on $\mathbb{R}$.

Then $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P(z)}{Q(z)}$$

where $Z_Q^+ = \{\text{zeroes of } Q \text{ in the } \underbrace{\text{UHP}}_{\text{Upper Half Plane}}\}$

Pf:



Z_Q^+

R big enough that all zeroes of Q in Z_Q^+ are inside the $\frown$.

$I_R : z(t) = t \,,\ -R \le t \le R . \qquad z'(t) = 1$

$$\int_{I_R} \frac{P(z)}{Q(z)}\, dz = \int_{-R}^{R} \frac{P(t)}{Q(t)} \cdot 1 \, dt \quad \longleftarrow \text{want!}$$

<u>Claim</u>: $\displaystyle\int_{C_R} \frac{P(z)}{Q(z)}\, dz \longrightarrow 0 \quad \text{as } R \to \infty.$

<u>Key</u>:
$$a\,|z|^{\deg P} \le \underline{|P(z)| \le A\,|z|^{\deg P}}\,, \quad |z| > R_P$$
$$\underline{b\,|z|^{\text{dep} Q} \le |Q(z)|} \le B\,|z|^{\deg Q}\,, \quad |z| > R_Q$$

$$\left|\int_{C_R}\right| \le \left(\max_{C_R} \left|\frac{P(z)}{Q(z)}\right|\right) \cdot \underbrace{\text{Length}(C_R)}_{\pi R}$$

$$\le \frac{A\, R^{\deg P}}{b\, R^{\deg Q}} \qquad \text{when } R > \text{Max}(R_P, R_Q)$$

$$\le \pi \frac{A}{b} \frac{1}{R^{\deg Q - \deg P - 1}} \longrightarrow 0 \quad \text{as } R \to \infty$$

since $\deg Q - \deg P - 1 \ge 1$.

Residue thm completes the proof, let $R \to \infty$.

<u>Note</u>: $\displaystyle\left|\frac{P(x)}{Q(x)}\right| \le \frac{1}{|x|^2} \quad \text{for} \quad |x| > R > 1$

shows $\displaystyle\int_{-\infty}^{\infty}$ is absolutely convergent.

So $\displaystyle\int_{-R}^{R} \longrightarrow \int_{\mathbb{R}} \quad \text{as } R \to \infty.$

Computing residues:

EX: Find $I = \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

Roots: 4-th roots of -1: $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$ $\quad r_1, r_2, r_3, r_4$

in UHP

outside

$$I = 2\pi i\left[\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} + \operatorname{Res}_{e^{i3\pi/4}} \frac{1}{z^4+1}\right]$$

Hmmm: $z^4+1 = (z-r_1)(z-r_2)(z-r_3)(z-r_4)$

Near r_1: $\frac{1}{z^4+1} = \frac{1}{z-r_1}\left[\underbrace{\frac{1}{(z-r_2)(z-r_3)(z-r_4)}}\right]$

$H(z) = A_0 + A_1(z-r_1) + \cdots$

analytic

$\underbrace{H(r_1)}_{=A_0} = 1/(r_1-r_2)(r_1-r_3)(r_1-r_4)$ $\quad \neq 0$

$$= \frac{A_0}{z-r_1} + \Big(\underbrace{A_1 + A_2(z-r_1) + \cdots}_{\text{regular power series}}\Big)$$

Ugh! $\operatorname{Res}_{r_1} = A_0 = 1/(r_1-r_2)(r_1-r_3)(r_1-r_4)$

Thm: Suppose $f(z), g(z)$ analytic near a and $g(z)$ has a simple zero at a. [assume $f(a) \neq 0$]

Then $\operatorname{Res}_a \dfrac{f(z)}{g(z)} = \dfrac{f(a)}{g'(a)}$

Pf: $g(z) = \underbrace{a_0}_{=g(a)} + \underbrace{a_1}_{\frac{g'(a)}{1!} \neq 0}(z-a) + \underbrace{a_2}_{\frac{g''(a)}{2!}}(z-a)^2 + \cdots$

$$= (z-a)\Big[\underbrace{a_1 + a_2(z-a) + \cdots}_{G(z) \text{ analytic near } a}\Big]$$

Notice $G(a) = a_1 = g'(a)/1!$
$G'(a) = a_2 = g''(a)/2!$

Now $$\frac{f(z)}{g(z)} = \frac{1}{z-a}\Big[\underbrace{\frac{f(z)}{G(z)}}_{\text{analytic}}\Big]$$

$A_0 + A_1(z-a) + \cdots$

$$= \frac{A_0}{z-a} + \big(\underbrace{A_1 + A_2(z-a) + \cdots}_{\text{regular power series}}\big)$$

So $$\operatorname{Res}_a \frac{f(z)}{g(z)} = A_0 = \left.\frac{f(z)}{G(z)}\right|_{z=a} = \frac{f(a)}{G(a)} = \frac{f(a)}{g'(a)} \checkmark$$

Problem: Find a similar formula for $\operatorname{Res}_a \dfrac{f(z)}{g(z)}$ when g has a double zero at a.

Back to $$I = 2\pi i\left[\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} + \operatorname{Res}_{e^{i3\pi/4}} \frac{1}{z^4+1}\right]$$

$$\frac{1}{z^4+1} = \frac{f(z)}{g(z)} \begin{matrix} \leftarrow 1 \\ \leftarrow z^4+1 \end{matrix}$$

$$g(e^{i\pi/4}) = 0$$

$$g'(e^{i\pi/4}) = 4(e^{i\pi/4})^3 \neq 0$$

zero of order 1 : simple

$$= 2\pi i \left(\frac{1}{4e^{i3\pi/4}} + \frac{1}{4e^{i9\pi/4}} \right)$$

$$= \frac{\pi i}{2} \left(\underbrace{e^{-i3\pi/4}}_{-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} + \underbrace{e^{-i9\pi/4}}_{\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} \right)$$

$$= \frac{\pi i}{2} \cdot \frac{-2i}{\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$