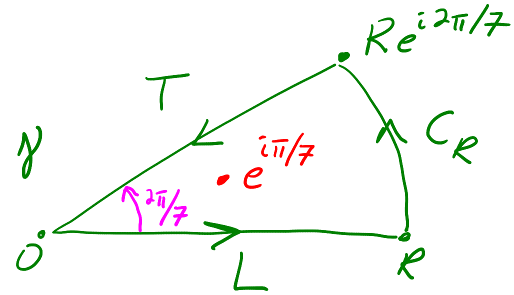
Review for Exam 1



3. Compute

$$\int_0^\infty \frac{t}{t^7+1}\,dt$$

by integrating a complex function around a "piece of pie" with vertex angle $\frac{2\pi}{7}$ and taking limits. If you claim a limit exists, prove it.

$$-T : z(t) = t e^{i2\pi/7}, \quad 0 \le t \le R \qquad z'(t) = e^{i2\pi/7}$$

$$\int_T \frac{z}{z^7+1}\,dz = -\int_{-T} = -\int_0^R \frac{t e^{i2\pi/7}}{t^7 \underbrace{e^{i2\pi}}_{=1} + 1}\left[e^{i2\pi/7}\right] dt$$

$$= -e^{i4\pi/7}\int_0^R \frac{t}{t^7+1}\,dt$$

$$\boxed{|z^7+1| \ge \big|\underbrace{|z^7|-1}_{R^7-1}\big| \quad \text{when } |z|=R>1}$$

$$\left|\int_{C_R}\right| \le \underbrace{\left(\max_{C_R}\left|\frac{z}{z^7+1}\right|\right)}_{\le \frac{R}{R^7-1}} \underbrace{\text{Length}(C_R)}_{\frac{2\pi}{7}R}$$

$$\le \frac{2\pi}{7}\frac{R^2}{R^7-1} \to 0 \text{ as } R\to\infty.$$

$$\int_L + \int_T + \int_{C_R} = 2\pi i \operatorname{Res}_{e^{i\pi/7}} \frac{z}{z^7+1} \begin{matrix} \leftarrow f(z) \\ \leftarrow g(z) \end{matrix}$$

$$g(e^{i\pi/7}) = 0$$

$$g'(e^{i\pi/7}) = 7e^{i6\pi/7} \ne 0$$

simple zero

$$\left(1 - e^{i4\pi/7}\right)\int_0^R \frac{t}{t^7+1}\,dt + \int_{C_R} =$$

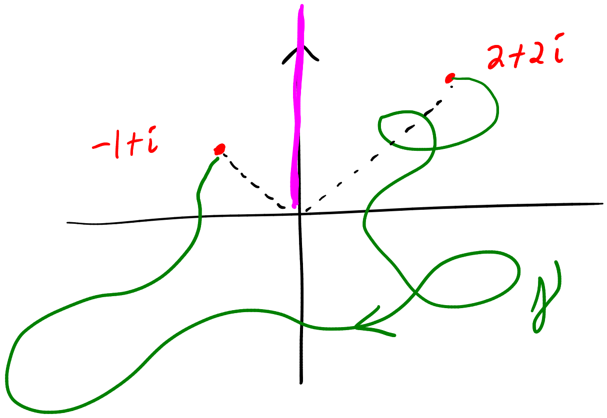
$$\left(1 - e^{i4\pi/7}\right) I + 0 = 2\pi i \frac{\left(e^{i\pi/7}\right)}{7\left(e^{i\pi/7}\right)^6} = \frac{2\pi i}{7e^{i5\pi/7}}$$

Solve for I.

1. Compute

$$\int_\gamma \frac{1}{z}\,dz$$

where γ is any curve in the plane that starts at $2+2i$ and ends at $-1+i$ and that avoids the set $\{z : z = it, t \geq 0\}$ (i.e., the positive imaginary axis, including $z=0$).



$\log_{\frac{\pi}{2}} z = \ln z + i\theta$

$\theta \in \arg z$

$\frac{\pi}{2} < \theta < \frac{5\pi}{2}$

$$\int_\gamma \frac{1}{z}\,dz = \log_{\frac{\pi}{2}}(-1+i) - \log_{\frac{\pi}{2}}(2+2i)$$

$$= \left(\ln\sqrt{2} + i\frac{3\pi}{4}\right) - \left(\ln 2\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi\right)\right)$$

$$= -\ln 2 - i\,3\pi/2$$

4. Suppose that f is an entire function such that

$$\lim_{|z|\to\infty} |f(z)| = \infty.$$

Non-vanishing $|z|>R$.

All zeroes in $|z|\le R$.

a) Explain why f can have at most finitely many zeroes.

b) Prove that f must be a non-constant polynomial.

Hint: The in-class proof of partial fractions.

Can't have lim pt. So finitely many.

$\frac{1}{f} - \sum^{N} R_i$ princ parts at poles $\equiv 0$

f rational! entire $\Rightarrow$ poly.

1. (20 pts) Given a complex polynomial $P(z)$ of degree $N \ge 1$, there exist real constants $0 < a < A$ and $R > 0$ such that the *basic polynomial estimate*,

$$a|z|^N \le |P(z)| \le A|z|^N$$

holds for $|z| > R$. Show that, if an entire function $f(z)$ satisfies the right hand side of the basic polynomial estimate, it must be a polynomial of degree N or less, and if it satisfies the left hand side, it must be a polynomial of degree N or more.

Cauchy estimates!

$$\left|f^{(n)}(0)\right| \le \frac{n!\,M}{r^n} \qquad M = \max_{|z|=r} |f(z)|$$

$$\le \frac{n!\,A\,r^N}{r^n} \quad \text{when } r > R$$

$$\le \frac{n!\,A}{r^{n-N}} \longrightarrow 0 \text{ as } r \to \infty \text{ when } n > N$$

Taylor's on $\mathbb{C}$: $f(z) = a_0 + a_1 z + \cdots + a_N z^N + 0 + 0 + \cdots$

poly! deg N or less.

Know $f(z)$ poly, say deg M.

$$\begin{cases} a|z|^M \le |f(z)| \le A|z|^M & |z|>R_1 \\ b|z|^N \le |f(z)| & |z|>R_2 \end{cases}$$

Conclude $M \ge N$.

4. (20 pts) Let C denote the unit circle parameterized in the counterclockwise sense. Given ϵ with $0 < \epsilon < 1$, compute

$$\int_C \frac{1}{z^2 + \epsilon z - \underline{\underline{1}}}\, dz.$$

$$z^2 + \varepsilon z - 1 = (z - r_1)(z - r_2) = z^2 \underbrace{-(r_1 + r_2)}_{\varepsilon} z + \underbrace{r_1 r_2}_{\substack{\shortparallel \\ -1}}$$

Roots : $\dfrac{-\varepsilon \pm \sqrt{\varepsilon^2 + 4}}{2} = r_1 r_2$

$4 < \varepsilon^2 + 4 < 5$

$2 < \sqrt{\varepsilon^2+4} < \sqrt{5}$

$1 < \dfrac{\sqrt{\varepsilon^2+4}}{2} < \dfrac{\sqrt{5}}{2}$

$$\underbrace{\overbrace{-\frac{\varepsilon}{2} - \frac{\sqrt{\varepsilon^2+4}}{2}}^{r_1}}_{\text{outside } C_1(0)} < -1$$

See $0 < r_2 < 1$.

$$I = 2\pi i \operatorname{Res}_{r_2} \frac{1}{z^2 + \varepsilon z - 1} = 2\pi i \frac{1}{2r_2 + \varepsilon}$$

$$= \frac{2\pi i}{2\left(-\frac{\varepsilon}{2} + \frac{\sqrt{\varepsilon^2+4}}{2}\right) + \varepsilon}$$

$$\int_C \frac{A}{z - r_1} + \frac{B}{z - r_2}\, dz$$

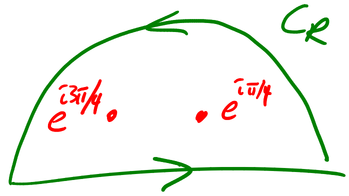
$$0 + B\, 2\pi i.$$

5. $$\int_{-\infty}^{\infty} e^{ist} \frac{1}{t^4 + 1}\, dt$$

Need $e^{isz} = e^{is(x+iy)} = \underbrace{e^{-sy}}_{r} \underbrace{e^{isx}}_{e^{i\theta}}$

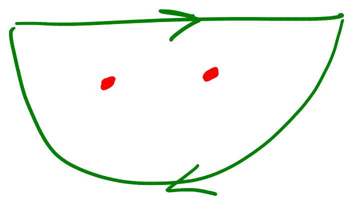
$$\left|e^{isz}\right| = e^{-sy} \begin{cases} < 1 & s > 0,\ y > 0 \\ < 1 & s < 0,\ y < 0 \end{cases}$$

$s > 0$:



$$\hat{f}(s) = 2\pi i \left(\operatorname{Res}_{e^{i\pi/4}} \frac{e^{isz}}{z^4+1} + \operatorname{Res}_{e^{i3\pi/4}} \frac{e^{isz}}{z^4+1} \right)$$

$s < 0$:



clockwise!

$$\hat{f}(s) = -2\pi i \left(\operatorname{Res}_{e^{-i\pi/4}} + \operatorname{Res}_{e^{-i3\pi/4}} \right)$$

$$\left| \int_{C_R} \right| \leq \left(\operatorname{Max} \frac{|e^{isz}|}{|z^4+1|} \right) \pi R \leq \frac{\pi R}{R^4 - 1}$$

$\longrightarrow 0$ as $R \to \infty$ in both cases.