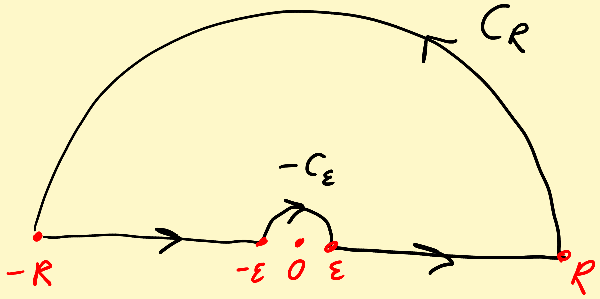
Lecture 22 More on the Residue thm

Exam 1 sol'ns posted on Home page
will return on Friday
HWK 5 due Friday in class



Jordan' lemma: Method to show

$$\left| \int_{C_R} \frac{e^{iz}}{z}\, dz \right| \to 0 \text{ as } R \to \infty.$$

Thm: P, Q polys

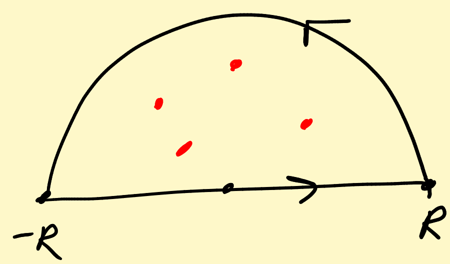
1) $\deg Q \geq \deg P + 1$ ← instead of $+2$.

2) Q has no zeroes on $\mathbb{R}$

Then
$$\lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} e^{ix}\, dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P(z)}{Q(z)} e^{iz}$$

where Z_Q^+ is set of zeroes of Q in the UHP.

Pf:



$$\left| \int_{C_R} \frac{P(z)}{Q(z)} e^{iz}\, dz \right| =$$

$$\left| \int_0^{\pi} \frac{P(Re^{it})}{Q(Re^{it})} e^{-R\sin t} e^{iR\cos t}\, iRe^{it}\, dt \right|$$

$$\leq \underbrace{\frac{A R^{\deg P}}{a R^{\deg Q}}}_{\text{from Basic poly ests}} R \underbrace{\int_0^{\pi} e^{-R\sin t}\, dt}_{\to 0 \text{ as } R\to\infty \text{ from last time}}$$

← Jordan's lemma

EX: $I = \int_0^{2\pi} \sin^4\theta \, d\theta =$

Hmmm: $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$C_1(0): z(\theta) = e^{i\theta}, \ 0 \le t \le 2\pi.$ Hmmm: $z'(\theta) = i e^{i\theta}$

$$I = \int_0^{2\pi} \left(\frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i} \right)^4 \frac{i e^{i\theta} \, d\theta}{\underbrace{i e^{i\theta}}_{iz(\theta)}} \quad dz$$

$$= \int_{C_1(0)} \left(\frac{z - \frac{1}{z}}{2i} \right)^4 \frac{1}{iz} \, dz$$

$$= 2\pi i \operatorname{Res}_0 \left(\frac{z - \frac{1}{z}}{2i} \right)^4 \frac{1}{iz}$$

$$= 2\pi i \operatorname{Res}_0 \left[\frac{1}{16} \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4} \right) \cdot \frac{1}{iz} \right]$$

$$= 2\pi i \cdot \frac{1}{16} \cdot 6 \cdot \frac{1}{i} = \frac{3\pi}{4}$$

Method: $R(z, w)$ rational

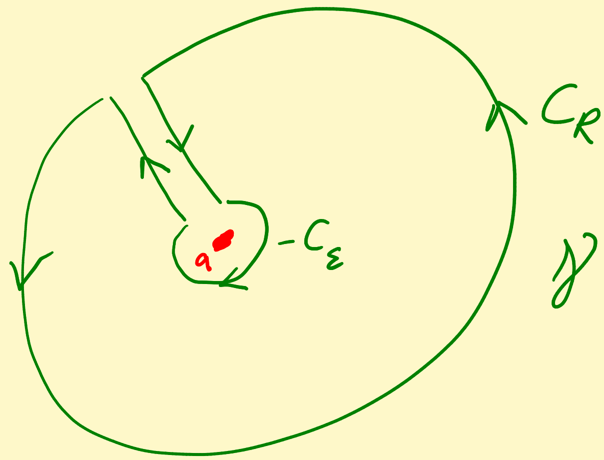
$$\int_0^{2\pi} R(\cos\theta, \sin\theta) \, d\theta$$

$$= \int_{C_1(0)} R\left(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i} \right) \frac{1}{iz} \, dz$$

Use Residue thm when no singularities on $C_1(0)$.

Keyhole trick

f analytic on D_{R+}



$$\int_\gamma \frac{f(z)}{z-a}\,dz = 0 \text{ by Cauchy thm for Toys.}$$

Step 1 Push sides together so $\int$s along lines cancel.

Step 2: $$\left(\int_{C_R} + \int_{-C_\varepsilon}\right)\frac{f(z)}{z-a}\,dz = 0$$

So
$$\int_{C_R}\frac{f(z)}{z-a}\,dz = \int_{C_\varepsilon}\frac{f(z)}{z-a}\,dz$$
$$= \int_0^{2\pi}\frac{f(a+\varepsilon e^{it})}{(a+\varepsilon e^{it})-a}\, i\varepsilon e^{it}\,dt$$
$$= i\int_0^{2\pi}\underbrace{f(a+\varepsilon e^{it})}_{\to f(a) \text{ as } \varepsilon\to 0}\,dt \longrightarrow i f(a)\, 2\pi$$

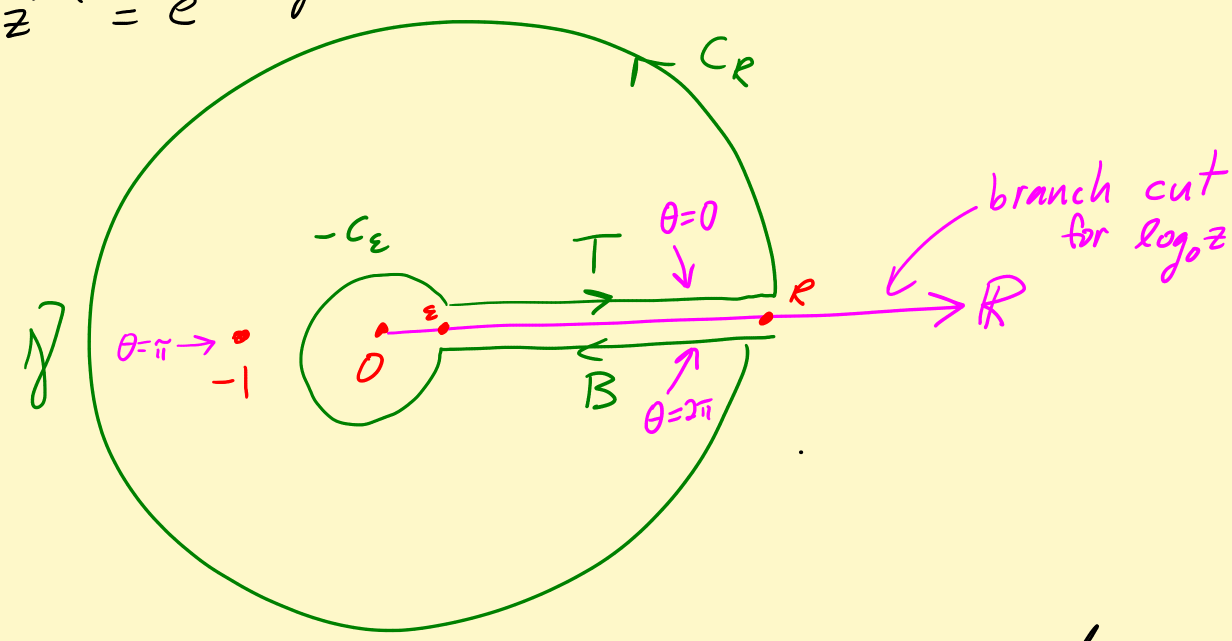
[f analytic near a $\Rightarrow$ f cont there]

Cauchy thm $\Longrightarrow$ Cauchy integral formula.

EX: $I = \int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx$, $0<\alpha<1$

$x^{-\alpha} = e^{-\alpha \ln x}$

Hmmm. $z^{-\alpha} = e^{-\alpha \log z}$ ← branch?



Residue thm on this Toy:

$$\int_\gamma \underbrace{\frac{e^{-\alpha \log z}}{z+1}}_{f(z)}\,dz = 2\pi i \operatorname{Res}_{-1} \frac{e^{-\alpha \log z}}{z+1}$$

$$= 2\pi i \frac{e^{-\alpha \log(-1)}}{1}$$

$$= 2\pi i\, e^{-\alpha(\ln|-1| + i\pi)}$$

$$= 2\pi i\, e^{-i\alpha\pi}$$

Limit as "mouth" closes:

$$\int_T f(z)\,dz = \int_\varepsilon^R \frac{e^{-\alpha(\ln t + i\cdot 0)}}{t+1}\,1\cdot dt$$

$$= \int_\varepsilon^R \frac{t^{-\alpha}}{t+1}\,dt \leftarrow \text{want}$$

$$\int_B f(z)\,dz = -\int_\varepsilon^R \frac{e^{-\alpha(\operatorname{Ln} t + i2\pi)}}{t+1}\, 1\cdot dt$$

$$= -\underset{\substack{\uparrow\\ \text{whew! They won't cancel out.}}}{e^{-i\alpha 2\pi}} \int_\varepsilon^R \frac{t^{-\alpha}}{t+1}\, dt$$

Show $\int_{C_R}$ and $\int_{C_\varepsilon} \to 0$ as $R \to \infty$ or $\varepsilon \to 0$.

$$|z^{-\alpha}| = |e^{-\alpha \log z}| = |e^{-\alpha(\operatorname{Ln}|z| + i\theta)}|, \quad |e^{i\theta}| = 1$$

$$= e^{-\alpha \operatorname{Ln}|z|} = |z|^{-\alpha}$$

$$\left|\int_{C_R}\right| \le \frac{R^{-\alpha}}{R-1}\cdot 2\pi R \to 0 \text{ as } R\to\infty \text{ because } \alpha > 0.$$

$$\left|\int_{C_\varepsilon}\right| \le \frac{\varepsilon^{-\alpha}}{1-\varepsilon}\cdot 2\pi\varepsilon \longrightarrow 0 \text{ as } \varepsilon \searrow 0 \text{ because } \alpha < 1.$$

$$C_\varepsilon: \quad |z-1| \ge \underbrace{\big||z|-1\big|}_{\substack{=1-\varepsilon \\ \text{when } |z|=\varepsilon<1}}$$

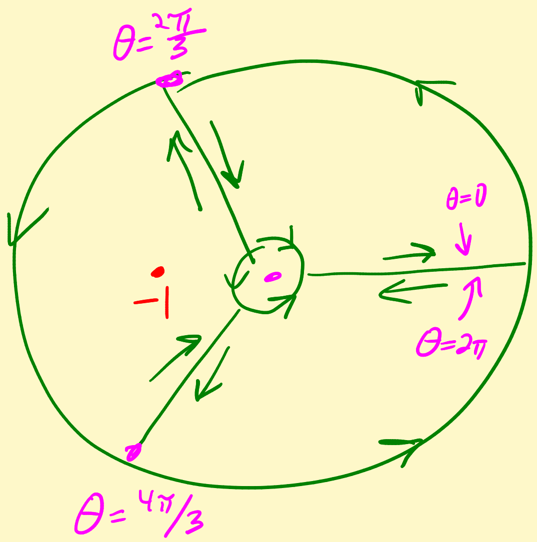
First close mouth. Then let $R\to\infty$. Then let $\varepsilon \searrow 0$. Get

$$\text{Res thm: } \Big(\underbrace{\int_T + \int_B}_{(1-e^{-i\alpha 2\pi})\int_\varepsilon^R \frac{t^{-\alpha}}{t+1}dt} + \underset{\searrow 0}{\int_{C_R}} + \underset{\downarrow 0}{\int_{-C_\varepsilon}}\Big) f(z)\,dz = 2\pi i\, e^{-i\alpha\pi}$$

Get $\quad I = \dfrac{2\pi i \, e^{-i\alpha\pi}}{1 - e^{-i\alpha 2\pi}} \cdot \dfrac{e^{i\alpha\pi}}{e^{i\alpha\pi}}$

$$= \pi \, \frac{2i \cdot 1}{e^{i\alpha\pi} - e^{-i\alpha\pi}} = \frac{\pi}{\sin \alpha\pi}$$

Or



Different branch log in each piece.
Same result