Lecture 23 Why the "Argument Principle"

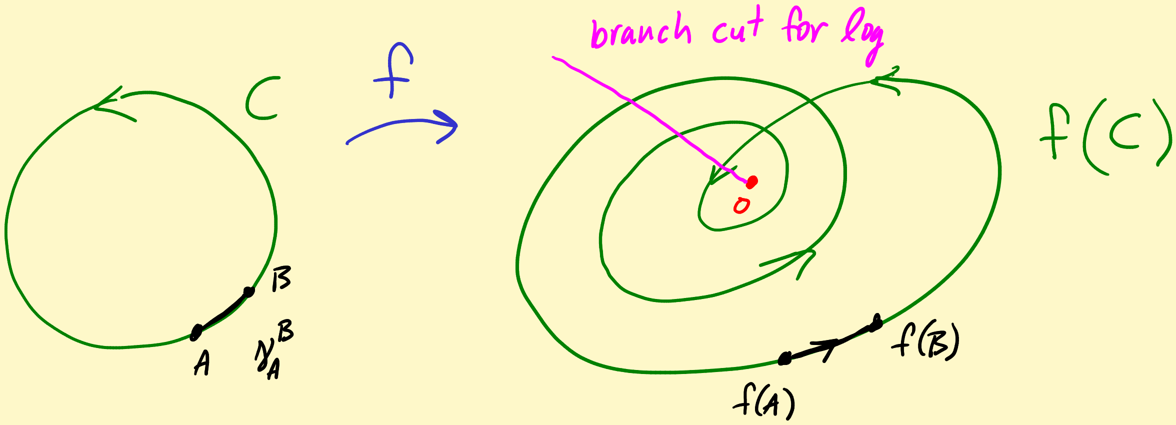
f analytic on $D_R(z_0)$, non-vanishing on $C_r(z_0)$, $0<r<R$.

$$I = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \begin{pmatrix} \#\text{ zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with multiplicity} \end{pmatrix}$$

If f allowed to have finitely many poles in $C_r(z_0)$ too,

then $$I = \begin{pmatrix} \#\text{ zeroes} \\ f \text{ in } C_r(z_0) \\ \text{with mult.} \end{pmatrix} - \begin{pmatrix} \#\text{ poles of } f \\ \text{inside } C_r(z_0) \\ \text{counted with order} \end{pmatrix}$$

Pf: $$\operatorname{Res}_a \frac{f'}{f} = \begin{cases} N & a \text{ is a zero of mult } N \\ -N & a \text{ is a pole of order } N \end{cases}$$



$$\int_{\gamma_A^B} \frac{f'}{f}\,dz = \int_{\gamma_A^B} \frac{d}{dz}\left[\log f(z)\right] dz$$

$$= \log f(B) - \log f(A)$$

$$= \left(\operatorname{Ln}|f(B)| + i \arg f(B)\right) - \left(\operatorname{Ln}|f(A)| + i\arg f(A)\right)$$

$$= \left(\mathrm{Ln}|f(B)| - \mathrm{Ln}|f(A)| \right) + i \underbrace{\Delta_{\gamma_A^B} \arg f}_{\text{does not depend on choice of cut!}}$$

Aha! Add up $\int$'s along little segments.

$\mathrm{Ln}|f(z_k)|$'s pairwise cancel, first cancels last!

$$\int_C \frac{f'}{f}\,dz = i \sum \Delta_{\gamma_k} \arg f = i \,\Delta_C \arg f$$

Divide by $2\pi i$: $\displaystyle \frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \left(\begin{array}{l} \#\text{ times } f(z) \\ \text{spins around } 0 \\ \text{in counterclockwise} \\ \text{sense} \end{array} \right)$

as z goes around C in counterclockwise sense.

Special case: $f(z) = z - a$

Def^{n}: $\displaystyle \mathrm{ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$

= "winding number of γ with respect to a"

= # times γ winds around a in the counterclockise sense.

Fact: $\gamma : z(t)$, $a \le t \le b$

$f(\gamma)$: $f(z(t))$, $a \le t \le b$. $\frac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)$!

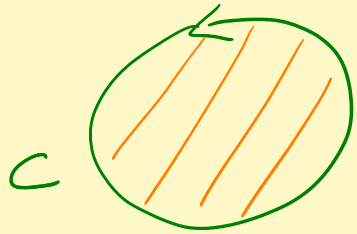
Arg princ: $F(z) = f(z) - w$

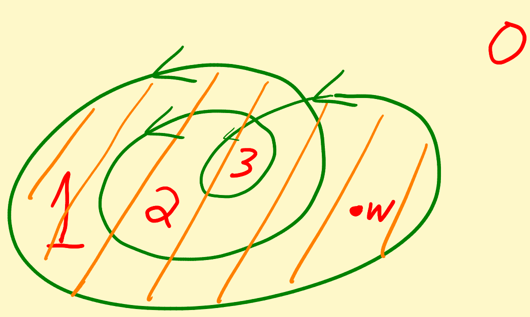
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-w}\, dz = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))}{f(z(t))-w}\, z'(t)\, dt$$

$$= \frac{1}{2\pi i}\int_{f(\gamma)} \frac{1}{\zeta - w}\, d\zeta$$

$$= \operatorname{ind}_{f(\gamma)}(w) = \text{winding } \#$$

Picture





Qual prob: How many zeroes does $z^{2001} + 1999z + 1$ have in the first quadrant?