Lecture 24 Important consequences of the Schwarz lemma

Schwarz lemma: $f : D_1(0) \to D_1(0)$ analytic, $f(0) = 0$, then

A) $|f(z)| \le |z|$ all $z \in D_1(0)$, and

$|f'(0)| \le 1$.

B) If $|f(z)| = |z|$ for some $z \ne 0$, or if $|f'(0)| = 1$, then $f(z) \equiv \lambda z$, const λ with $|\lambda| = 1$.

HWK 1, prob 1: $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, $a \in D_1(0)$ (Möbius transf)

Facts: φ_a analytic, $\varphi_a : D_1(0) \circlearrowleft \underset{\text{onto}}{\overset{1\text{-}1}{}}$, $\varphi_a^{-1} = \varphi_{-a}(z) = \dfrac{z+a}{1+\bar{a}z}$

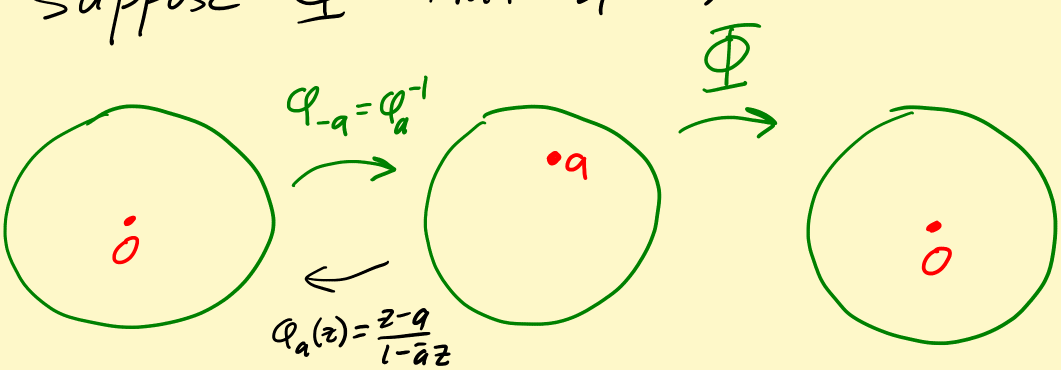
Def^n: $\underset{\text{domain}}{Aut(\Omega)} = \{ f : \Omega \circlearrowleft \underset{\text{onto}}{\overset{1\text{-}1}{}},\ \text{analytic} \}$ ← f biholomorphic

Fact: $Aut(\Omega)$ is a group under composition.

[Need "Super inverse fcn thm: f analytic, 1-1, onto $\implies$ f' non-vanishing & f^{-1} analytic.]

Thm: $Aut(D_1(0)) = \{ \lambda \varphi_a : a \in D_1(0),\ |\lambda| = 1 \}$

Pf: Suppose $\Phi \in Aut(D_1(0))$



Lemma: If $\Psi \in Aut(D_1(0))$ with $\Psi(0)=0$, then $\Psi(z) \equiv \lambda z$ for const λ with $|\lambda|=1$.

Pf: Apply Schwarz to Ψ : $|\Psi'(0)| \le 1$

Can apply Schw to Ψ^{-1} too : $\underbrace{\left|(\Psi^{-1})'(0)\right|}_{\frac{1}{|\Psi'(\underbrace{\Psi^{-1}(0)}_{=0})|}} \le 1$

So $|\Psi'(0)| \ge 1$.

So $|\Psi'(0)| = 1$. Sch, part 2 $\Rightarrow \Psi(z) \equiv \lambda z$ ✓

Back to proof: $\Psi(z) = \Phi(\underbrace{\varphi_{-a}(z)}_{w}) = \lambda z$

$z = (\varphi_{-a})^{-1}(w) = \varphi_a(w)$

So $\Phi(w) = \lambda \varphi_a(w)$ ✓

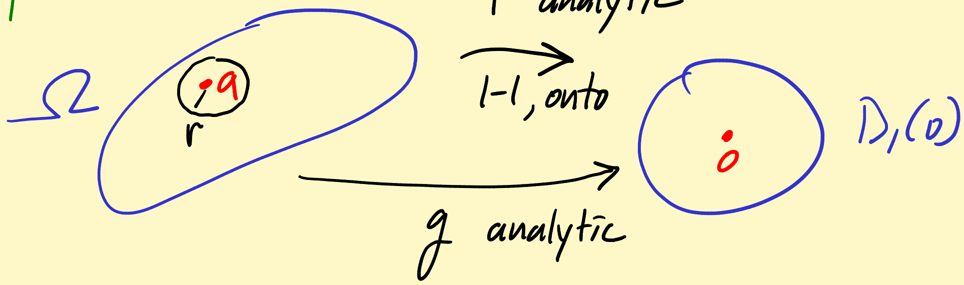
Fact: $Aut(D_1(0))$ is transitive : Given $a, b \in D_1(0)$, $\exists\ \varphi \in Aut(D_1(0))$ with $\varphi(a) = b$. $\varphi_{-b} \circ \varphi_a$

Riemann mapping thm: $\Omega \not\subset \mathbb{C}$, simply connected domain (no holes). $\exists\ f: \Omega \to D_1(0)$ analytic 1-1, onto.

$Aut(\Omega)$:

$$\begin{tikzcd} \Omega \arrow[r, "\Phi"] & \Omega \arrow[d, "f"] \\ D_1(0) \arrow[u, "f^{-1}"] \arrow[r, "\lambda \varphi_a(z)"] & D_1(0) \end{tikzcd}$$

HWK 5 problem:



Schwarz on $g(F^{-1}(z))$ (like for Ψ above)

implies that $|g'(a)| \le |F'(a)|$.

<u>Preview of pf of Riemann mapping thm.</u>

$M = \sup \{ |g'(a)| : g : \Omega \to D_1(0) \text{ analytic}, g(a)=0 \}$.

Riemann map F : $|F'(a)| = M$.

<u>Step 1</u>: $M < \infty$: Cauchy est ; $|g'(a)| \le \frac{1}{r}$

$r \nearrow \text{dist}(a, b\Omega)$: $|g'(a)| \le \frac{1}{\text{dist}(a, b\Omega)}$

<u>Step 2</u>: Get seq $g_n : \Omega \to D_1(0)$, analytic, 1-1,

with $|g_n'(a)| \to M$.

<u>Step 3</u>: Can take subseq g_n converges uniformly on compact subsets of Ω to an analytic fcn f. (Montel's thm).

<u>Step 4</u>: See limit f is 1-1. (Hurwicz thm)

Hurwicz : limit of 1-1 analytic fcns is either 1-1 or constant.

Step 5: Finally, show f is onto: Schwarz!

So f is the Riemann map.

Linear fractional transformations (LFTs)

some books: Möbius transf.

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

Note: $ad-bc=0$, rows of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ lin dependent

and either dem $L(z) \equiv 0$ or

num $L(z) = c(\text{den } L(z))$. $L(z) \equiv$ const.

Facts. $L: \hat{\mathbb{C}} \xrightarrow[\text{onto}]{1-1} \hat{\mathbb{C}}$ $\qquad \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

Think: LFTs $= \mathrm{Aut}(\hat{\mathbb{C}})$.

$c \neq 0$: $\displaystyle\lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c} \qquad L(\infty) = \frac{a}{c}$

$z = -\frac{d}{c}$ is a simple pole of L: $L\left(-\frac{d}{c}\right) = \infty$

$L: \hat{\mathbb{C}} \circlearrowleft$ $\begin{matrix}1-1\\ \text{onto}\end{matrix}$ homeomorphism.

$c=0$: $L(z) = \frac{az+b}{d} = Az+b$: $L(\infty) = \infty$

$w = L(z) = \frac{az+b}{cz+d}$. Solve for z: $z = \frac{dw-b}{-cw+a}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Hmmm: $\mathbb{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad \mathbb{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Cool thing: $L \sim \mathbb{A}$ $\qquad id(z) = z \sim \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$L^{-1} \sim \mathbb{A}^{-1}$

$$\left.\begin{array}{l} L_1 \sim \mathbb{A}_1 \\ L_2 \sim \mathbb{A}_2 \end{array}\right\} \quad L_1 \circ L_2 \sim \mathbb{A}_1 \mathbb{A}_2$$

matrix mult!

Fact: $L \rightsquigarrow \mathbb{A}$ is a group homomorphism into group of 2×2 complex matrices with $\det \neq 0$.

See p. 248 of Taylor.