Lecture 25 LFTs, Jukovsky map, conformal mapping

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

Fact: LFTs : $\left\{\begin{matrix}\text{lines}\\ \text{circles}\end{matrix}\right\} \circlearrowleft$

Lemma: LFTs are generated by 4 basic maps:

1) $z \mapsto rz, \quad r \in \mathbb{R}^+$ (stretch)
2) $z \mapsto e^{i\theta} z$ (rotation)
3) $z \mapsto z + b, \quad b \in \mathbb{C}$ (translation)
4) $z \mapsto \frac{1}{z}$ (inversion)

Pf: Case $c=0$. $L(z) = Az + B$ 1,2,3 enough

$A = re^{i\theta}$

Case $c \neq 0$.
$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\left(b - \frac{ad}{c}\right)}{cz+d}$$

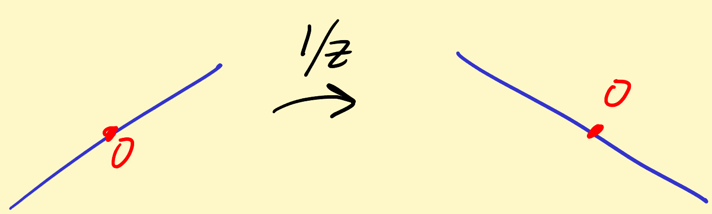
$\left(b - \frac{ad}{c}\right) = Re^{i\psi}$, $c = re^{i\theta}$

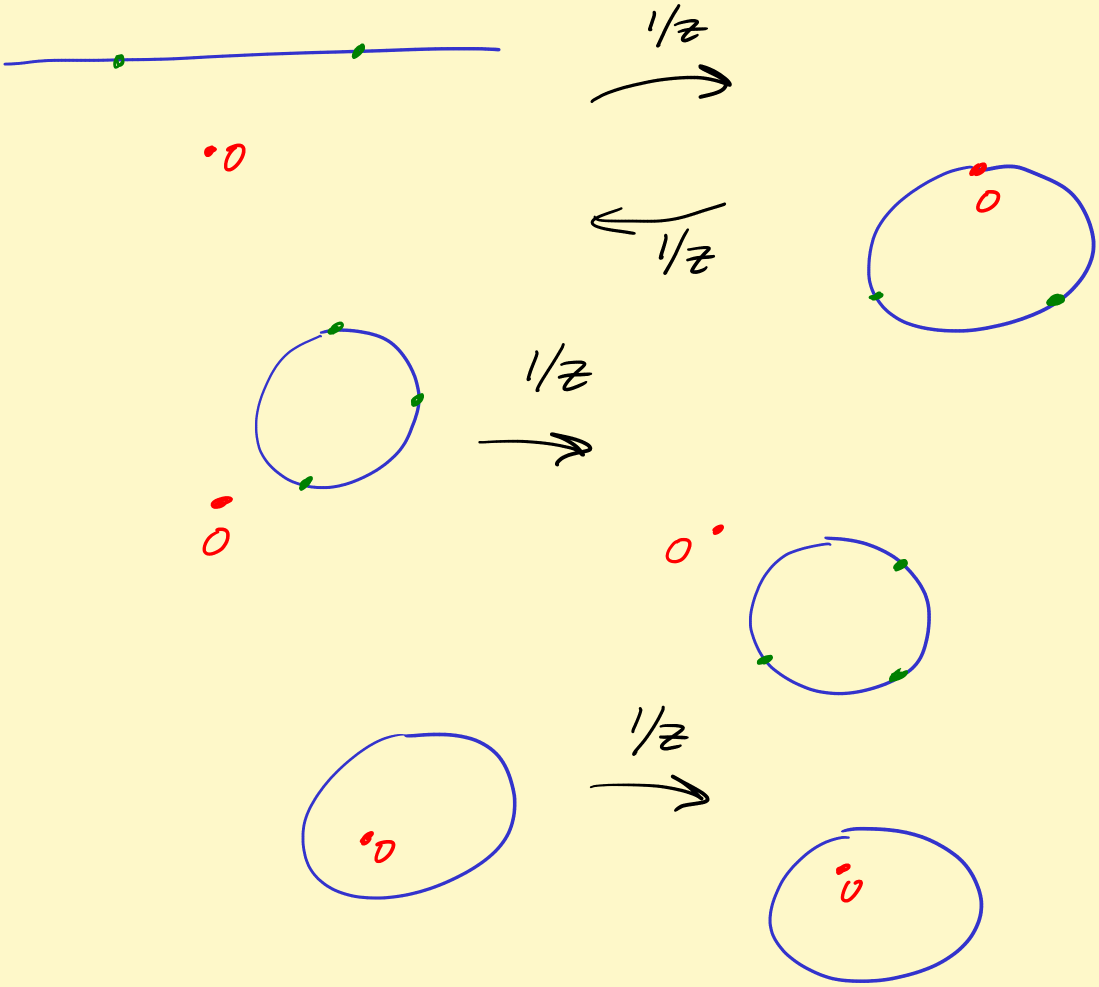
7 maps from 1–4!

Cor: LFTs : $\left\{\begin{matrix}\text{lines}\\ \text{circles}\end{matrix}\right\} \circlearrowleft$

Pf: 1, 2, 3 : $\{\text{lines}\} \circlearrowleft$, $\{\text{circles}\} \circlearrowleft$

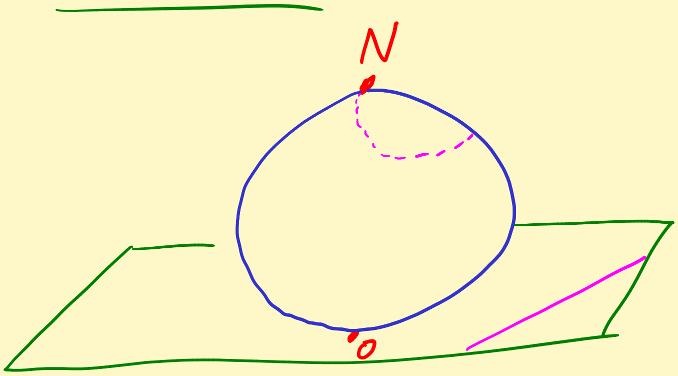
4 can mix them up





<u>Pf</u>: HS analytic geom

<u>Fun fact</u> On Riemann sphere $\left\{ \begin{matrix} \text{lines} \\ \text{circles} \end{matrix} \text{ in } \mathbb{C} \right\} \sim$ slices



Fact from last time: LFT : $\hat{\mathbb{C}} \xrightarrow[\text{onto}]{1-1} \hat{\mathbb{C}}$

<u>Lemma</u>: A LFT that fixes 3 distinct pts must be the identity.

Pf: $L(z_j) = z_j \qquad j=1,2,3$

$$\frac{az_j+b}{cz_j+d} = z_j$$

$$cz_j^2 + (d-a)z_j - b = 0$$

Quad eqn with 3 roots! Coeff must be 0s.

$c=0, \quad a=d \neq 0, \quad b=0$

$$L(z) = \frac{az}{d} = z \quad \checkmark$$

Cor: If two LFTs agree at 3 pts, they are the same.

3 pts determine a circle!

2 pts (plus the pt at ∞) determine a line.

Useful LFTs

$$\frac{z-a}{z-b} \qquad a \mapsto 0, \quad b \mapsto \infty$$

$$\left(\frac{c-b}{c-a}\right)\frac{z-a}{z-b} \qquad a \mapsto 0, \quad b \mapsto \infty, \quad c \mapsto 1$$

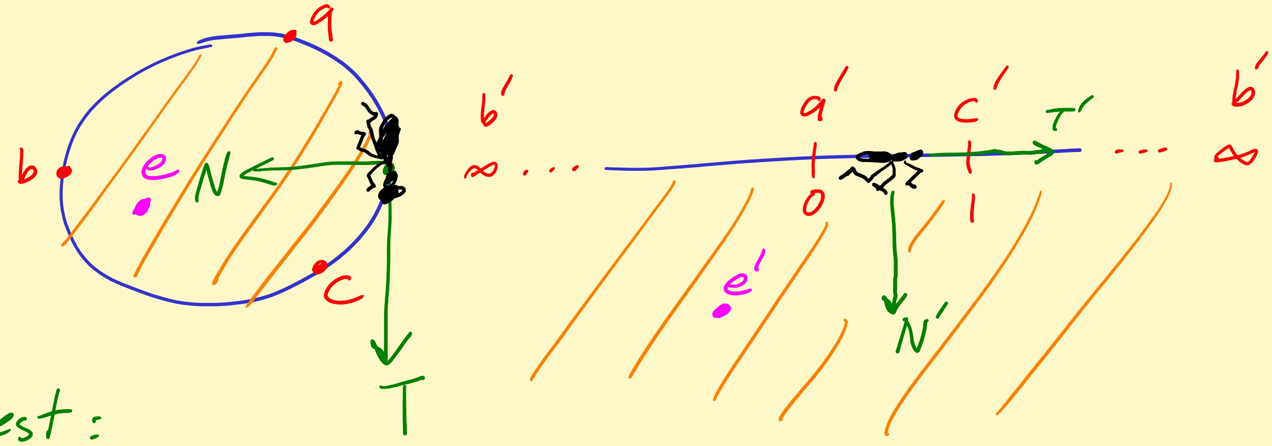
Prob: Find an LFT $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$

Sol^n:

$$\begin{array}{ccc} \xrightarrow{L_1} & & \xleftarrow{L_2} \\ z_1 \mapsto 0 & \longleftarrow & w_1 \\ z_2 \mapsto 1 & \longleftarrow & w_2 \\ z_3 \mapsto \infty & \longleftarrow & w_3 \end{array}$$

Sol^n: $L_2^{-1} \circ L_1$

EX:



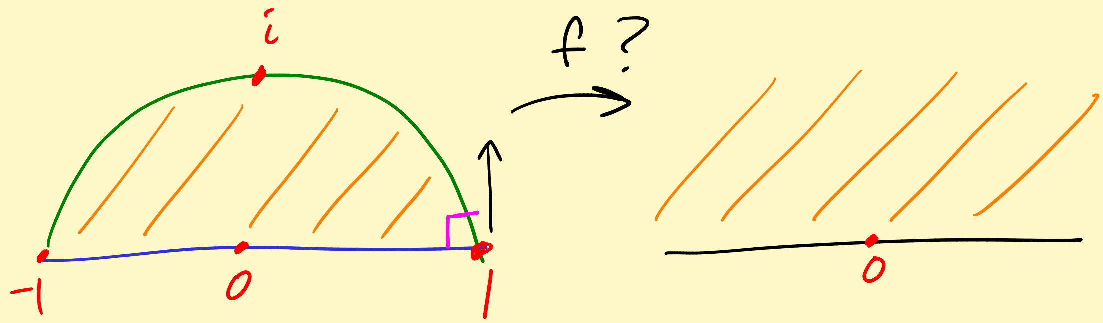
Bug test:

Use conformality to see which side gets mapped to which side

One pt test: Pick a point. See where it goes.

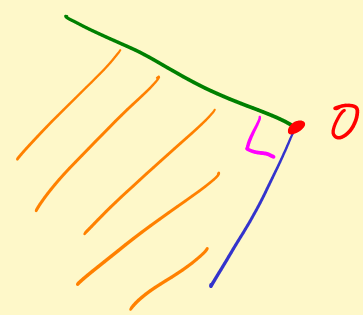
Use fact that $\{\text{Connected components of } \hat{\mathbb{C}} - \{\text{line or circle}\}\} \longleftrightarrow$ ditto

EX:

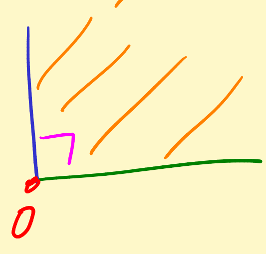


Trick: Blow a corner out to ∞ : e.g. $L(z) = \dfrac{z-1}{z-(-1)}$

L: Conformality :



Rotate:



z^2 :

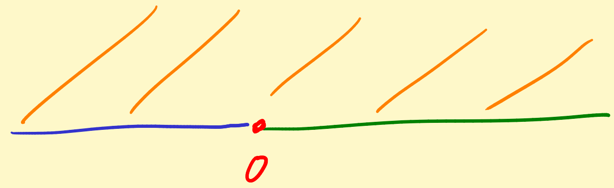


Figure it out.

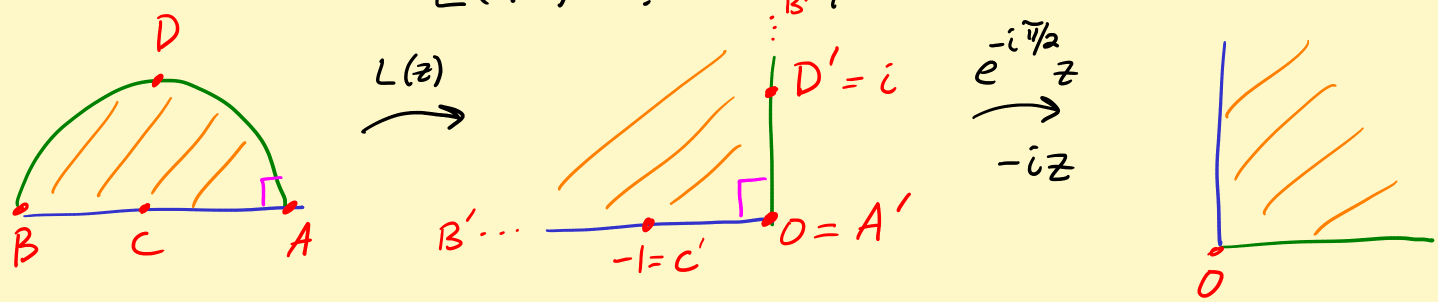
$$L(1) = 0 \qquad\qquad L(z) = \frac{z-1}{z+1}$$

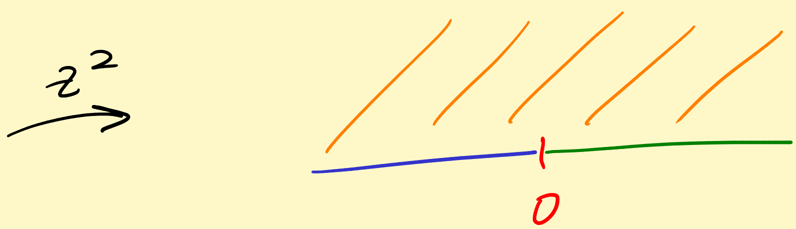
$$L(-1) = \infty$$

$$L(i) = \frac{i-1}{i+1} = i$$

$$L(0) = -1$$

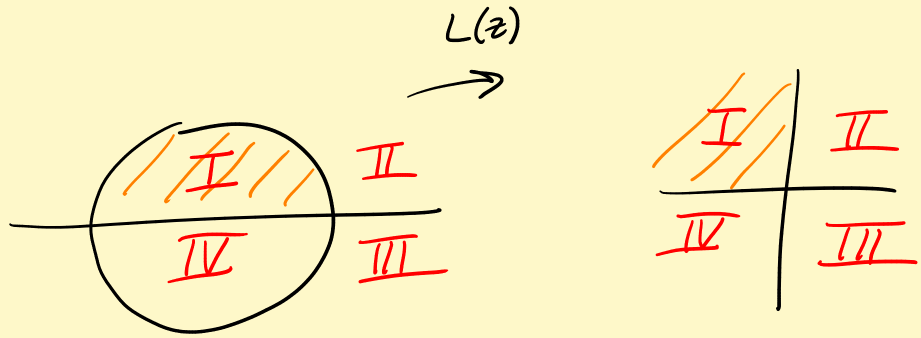
$$L(i/2) = \,?$$ one-pt test



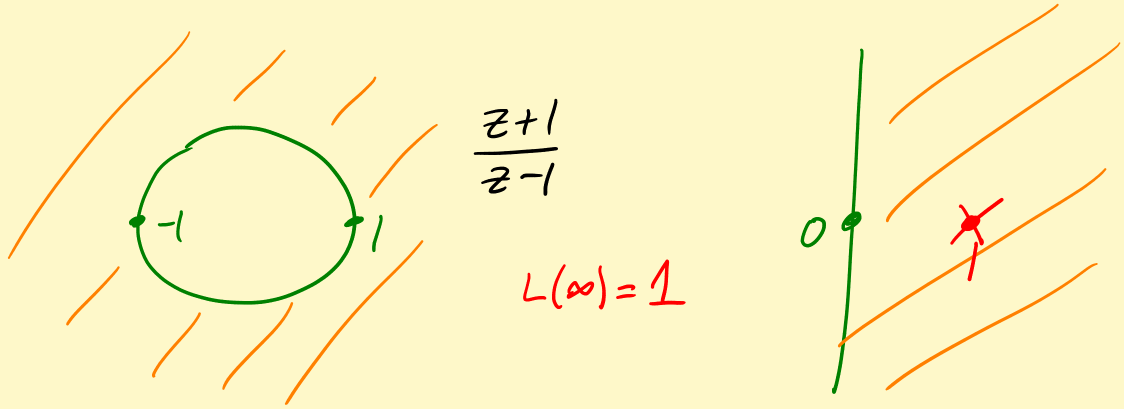


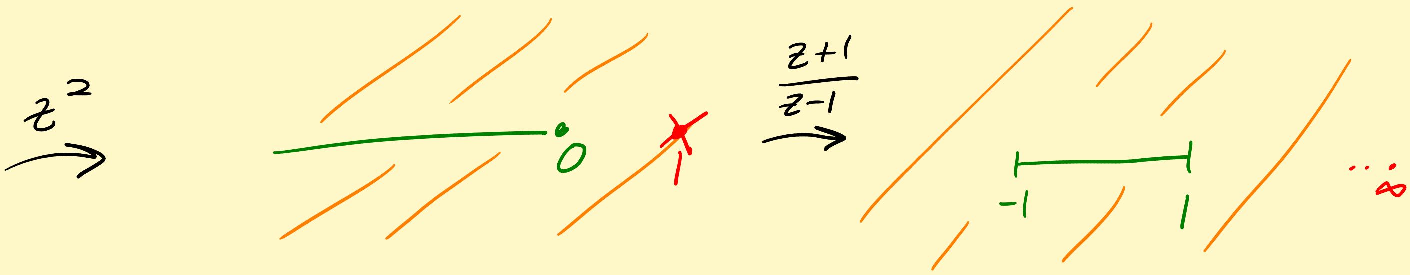
$$f(z) = \left(-i\left(\frac{z-1}{z+1}\right)\right)^2 = -\left(\frac{z-1}{z+1}\right)^2$$

Big picture



Cook up $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$





Composition $= J(z)$