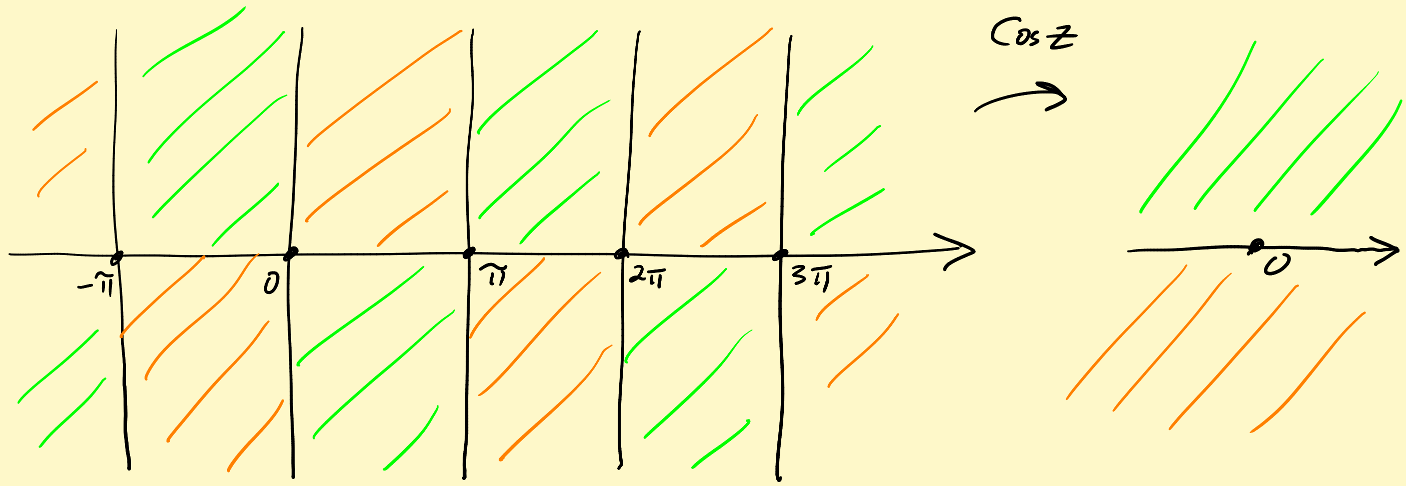
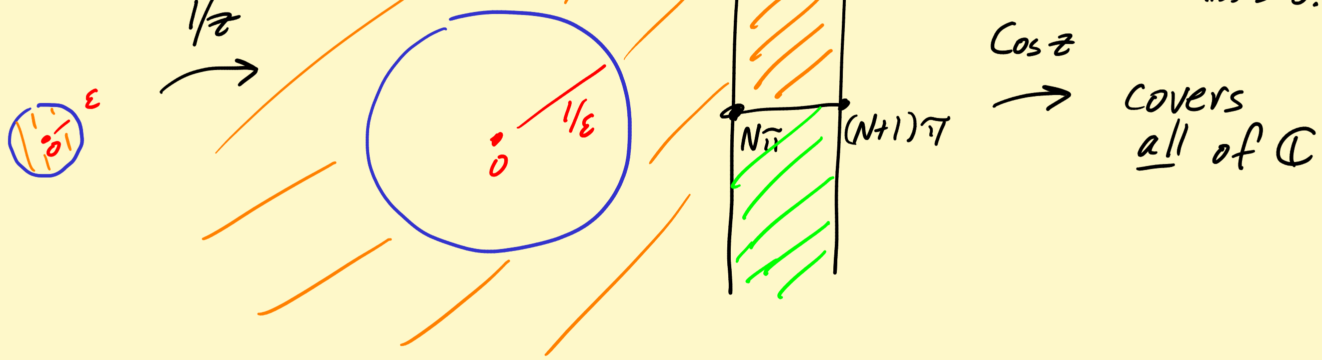
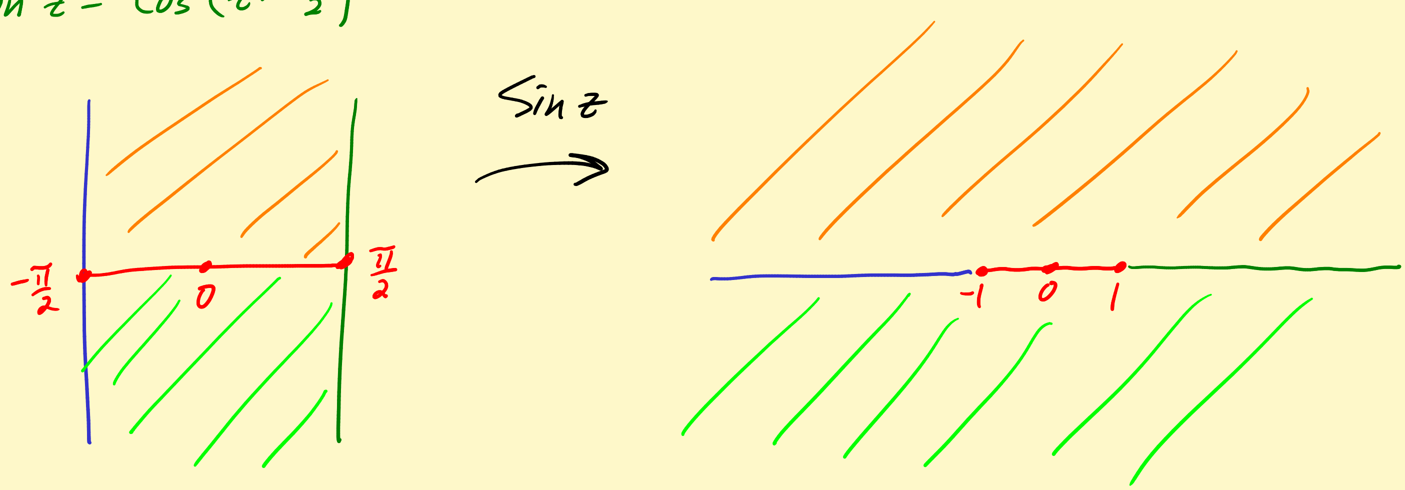
Lecture 26 Conformal maps, Schwarz reflection

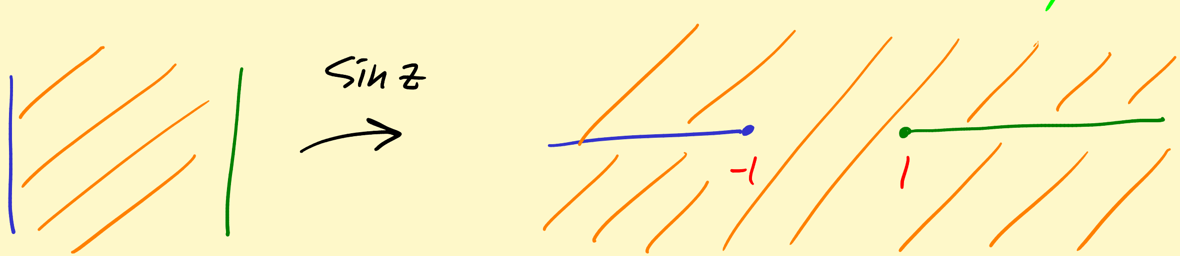


$\cos 1/z$ has an essential sing at $z=0$: maps $D_\varepsilon(0)-\{0\}$ onto $\mathbb{C}$ for each ε. (Unlike $e^{1/z}$ which misses 0.)

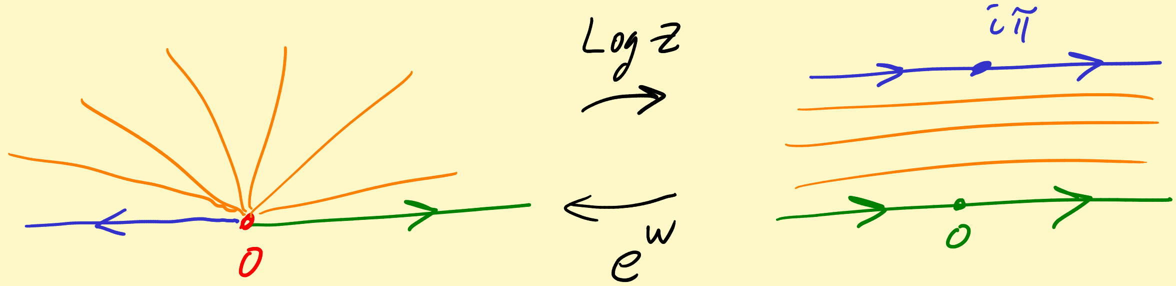


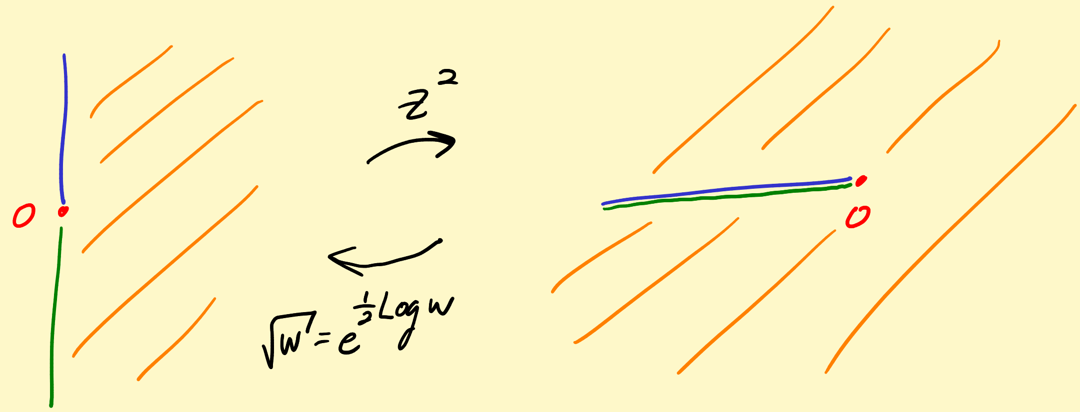
$\sin z = \cos\left(z - \frac{\pi}{2}\right)$

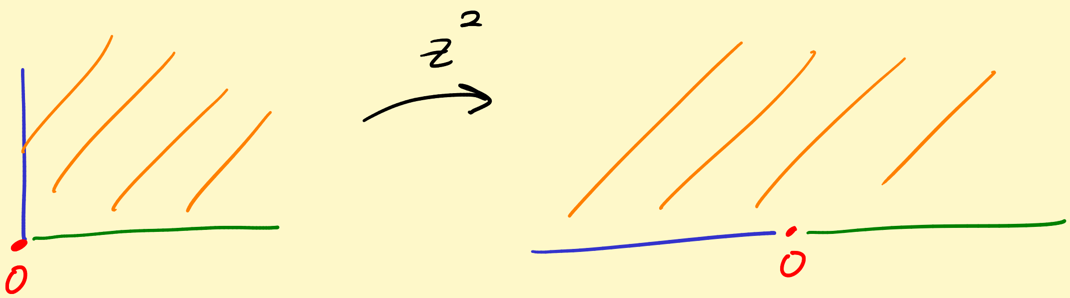


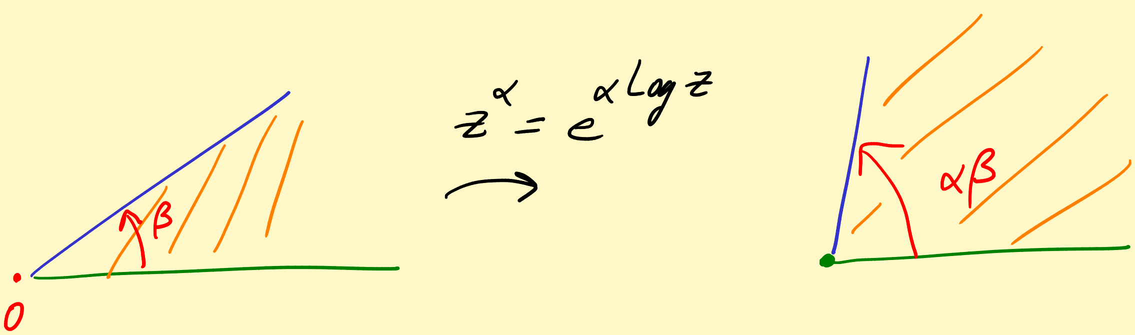


Other useful conformal maps

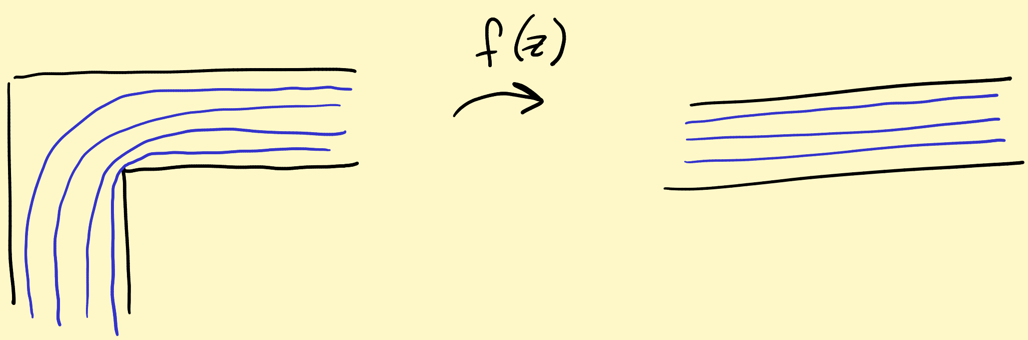




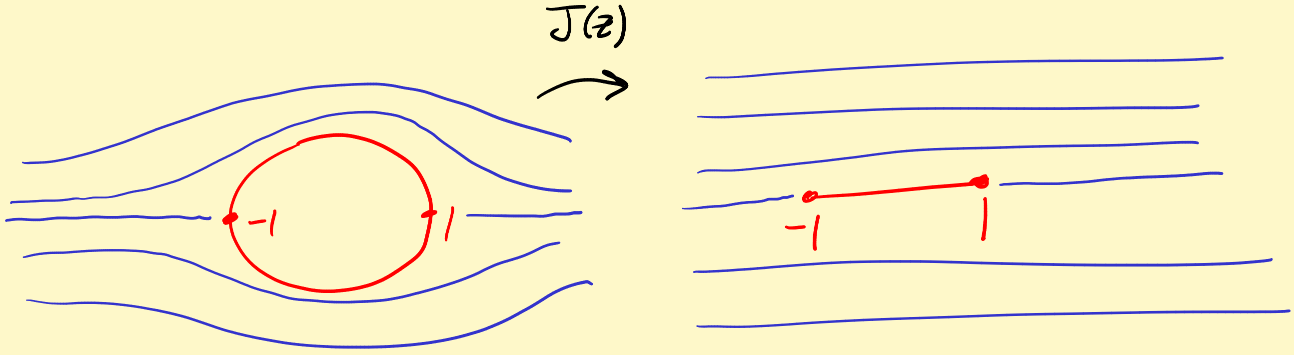




Engineers

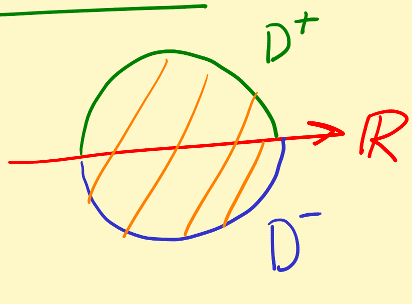


Jukovsky map



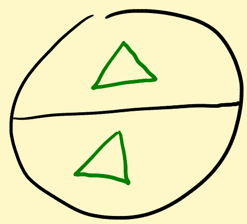
Schwarz reflection

Lemma

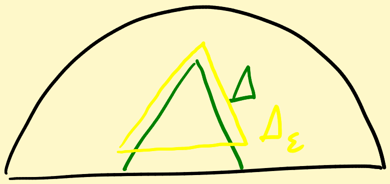


f continuous on D, analytic on D^+ and D^-. Then f analytic on D.

Pf: Morera's thm.

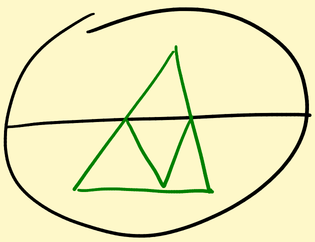


$$\int_{\Delta's} = 0$$



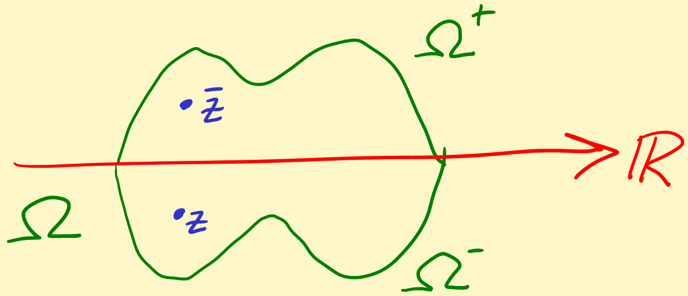
$$\Delta_\varepsilon : z_\Delta(t) + i\varepsilon$$

$$\int_\Delta f\,dz = \lim_{\varepsilon\to 0} \underbrace{\int_{\Delta_\varepsilon} f\,dz}_{=0} = 0$$



$$\int_\Delta = \text{Sum} \int_{\Delta's}$$

Schwarz reflection principle



f analytic in Ω^+, continuous up to $\mathbb{R}$, and real valued along $\mathbb{R}$.

Then

$$F(z) = \begin{cases} f(z) & \text{in } \Omega^+ \\ f(x) & x \in \mathbb{R} \\ \overline{f(\bar{z})}, & z \in \Omega^- \end{cases}$$

Pf: f real along $\mathbb{R}$, cont up to $\mathbb{R} \Rightarrow F$ cont on Ω.

F analytic on Ω^+, and on Ω^- by HWK 5 prob.

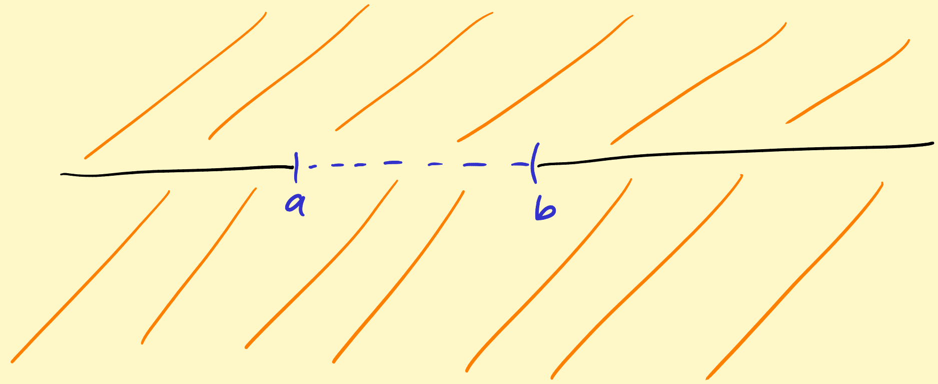
Lemma $\Rightarrow F$ analytic on Ω.

Remark: HWK 5 prob: f analytic $\Rightarrow \overline{f(\bar{z})}$ analytic

$$f(z) = f(x+iy) = u(x,y) + iv(x,y) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$\overline{f(\bar{z})} = \overline{f(x-iy)} = \underbrace{u(x,-y) - iv(x,-y)}_{\substack{\text{C-R eqns,} \\ \text{chain rule}}} = \underbrace{\sum_{n=0}^{\infty} \bar{a}_n (\bar{z}-\bar{z}_0)^n}_{\substack{R = \text{same} \\ \text{via Hadamard}}}$$

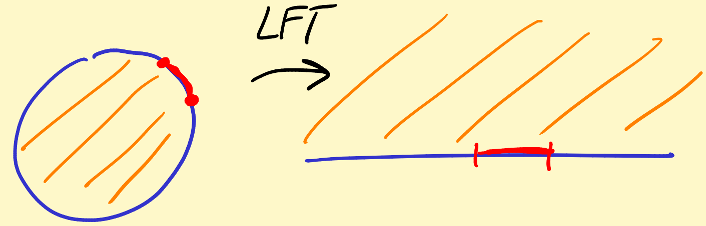
Consequence of Schwarz: Suppose f is analytic on UHP and continuous up to $(a,b) \subset \mathbb{R}$ and zero there. Then $f \equiv 0$ on UHP.



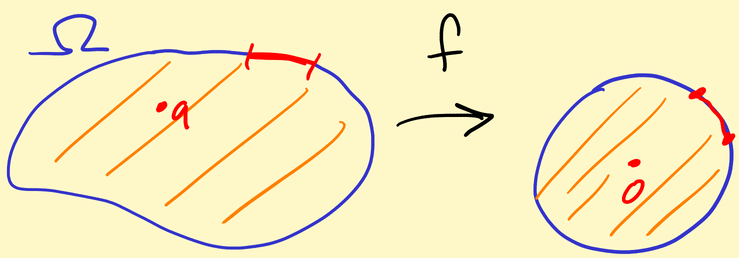
Get analytic reflection to $\mathbb{C} - \left((-\infty, a] \cup [b, \infty) \right)$

zeroes pile up at lim pts in (a,b). Identity thm $\Rightarrow f \equiv 0$.

<u>Tricks</u>: Use conformal mapping to deduce similar thm for othe domains: e.g.

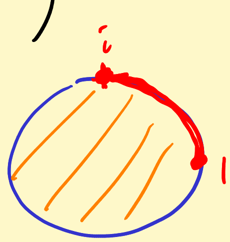


Riemann mapping thm



Hmmm. Ω bounded by a Jordan curve. Is Riemann map continuous up to the boundary? Yes! (Carathéodory.)

<u>Fun exercise</u>:



f analytic, cont up to $C_1(0)$, $f \equiv 0$ on red.

$$F(z) = f(z)\,f(iz)\,f(-z)\,f(-iz)$$

analytic on $D_1(0)$, cont on $\overline{D_1(0)}$, zero on $C_1(0)$.

Max princ $\Longrightarrow$ $F \equiv 0$.

A factor must vanish. (Hwk 5). So $f \equiv 0$.