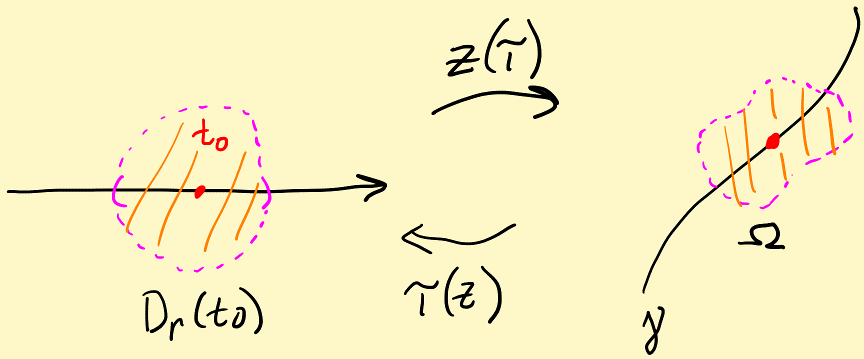
Lecture 27 Real analytic curves & Schwarz reflection, Laurent expansions

γ a real analytic curve: $z(t) = \sum_{n=0}^{\infty} c_n (t-t_0)^n$ convergent real p-series

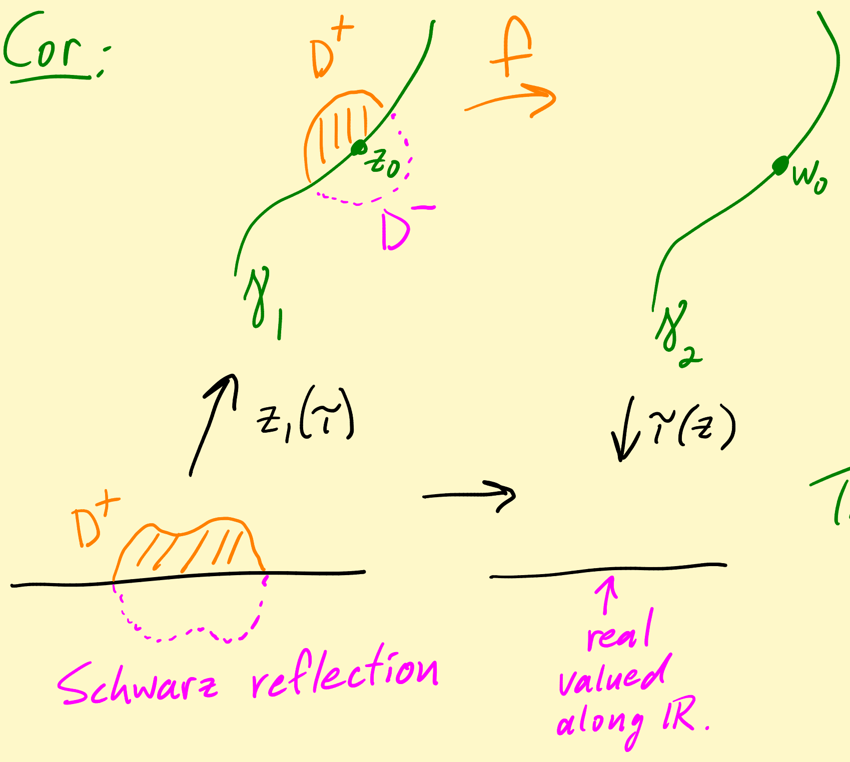
$z'(t_0) \neq 0$



Let t become complex $\tilde{\tau}$ on $D_r(t_0)$.

Can shrink r so that $z(\tilde{\tau})$ analytic, 1-1, onto domain Ω.

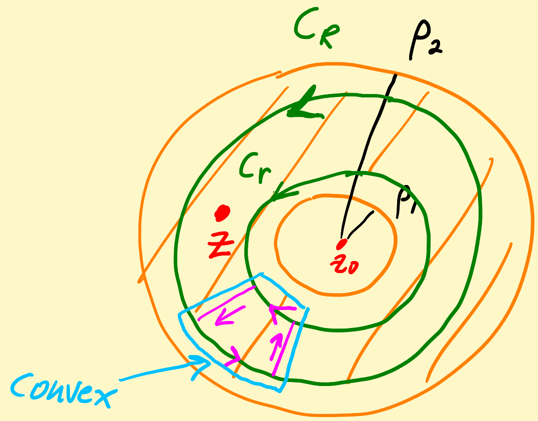
Cor:



γ_1, γ_2 real analytic curves. f analytic on D^+ and extends continuously to γ_1, and $f(\gamma_1) \subset \gamma_2$

Then f extends analytically to a nbhd of z_0.

Laurent expansions



f analytic on $A(\rho_1, \rho_2)$

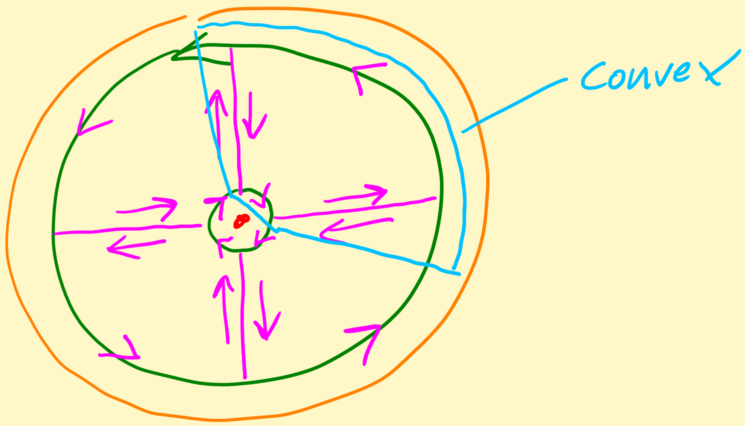
$0 < \rho_1 < r < R < \rho_2$

Cauchy theorem for an annulus: f analytic on $A(\rho_1, \rho_2)$.

$$\left(\int_{C_R} - \int_{C_r}\right) f \, dz = 0.$$

Pf: Can cut up into small pie pieces, add up counterclockwise integrals, use Cauchy on convex, note that integrals along cuts cancel pairwise, see result.

EX:



Cauchy integral thm

$$f(z) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(w)}{w-z}\, dw$$

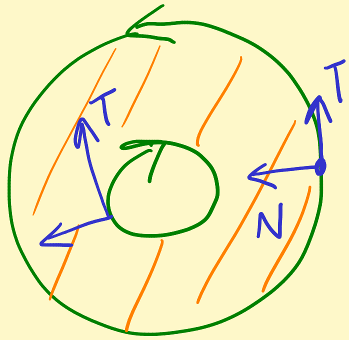
when $r < |z - z_0| < R$.

Pf: Apply Cauchy thm to $DQ = \dfrac{f(w) - f(z)}{w - z}$, which has a removable sing at $w=z$. So

$$0 = \left(\int_{C_R} - \int_{C_r}\right) DQ \, dz = \left(\int_{C_R} - \int_{C_r}\right) \frac{f(w)}{w-z}\, dw$$

$$- \underbrace{f(z)}_{\text{const}} \left(\underbrace{\int_{C_R} \frac{dw}{w-z}}_{2\pi i} - \underbrace{\int_{C_r} \frac{dw}{w-z}}_{0 \text{ by Cauchy thm}}\right)$$

Remark



"Standard orientation of the boundary"

Inward pointing normal vector to left of T-vector

Laurent series

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}}_{}$$

Converge in $D_{\rho_2}(z_0)$ absolutely and uniformly on $D_R(z_0)$ $\rho_1 < r < R < \rho_2$

converge in $\{z : |z-z_0| > \rho_1\}$, absolutely and uniformly on $\{z : |z-z_0| > r\}$ $\rho_1 < r < R < \rho_2$

$$a_n = \frac{1}{2\pi i} \int_{C_{\tilde{r}}} \frac{f(z)}{(z-z_0)^{n+1}} \, dz \longleftarrow n \in \mathbb{Z}$$

Cauchy thm $\Rightarrow$ indep of $\int_{C_r}$ with respect to r

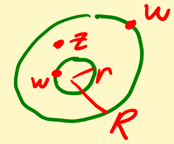
Theorem: Suppose f has an isolated sing at z_0.

1) $a_{-n} = 0$ for $n = 1, 2, 3, \ldots \iff z_0$ is removable

2) Only finitely many $a_{-n} \neq 0$, $(n \in \mathbb{N}) \iff z_0$ pole

3) Infinitely many $a_{-n} \neq 0$, $(n \in \mathbb{N}) \iff z_0$ essential.

Def^n: $a_{-1} = \operatorname{Res}_{z_0} f$ at isolated sing.



Pf of expansion Let $z_0=0$.

$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{\underbrace{w-z}_{w\left(1-\frac{z}{w}\right),\ \left|\frac{z}{w}\right|<1}}\,dw}_{\text{power series}} \;-\; \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{\underbrace{w-z}_{-z\left(1-\frac{w}{z}\right),\ \left|\frac{w}{z}\right|<1}}\,dw}_{\text{new negative terms}}$$

$$= \underbrace{\frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\left[1+\left(\frac{z}{w}\right)+\left(\frac{z}{w}\right)^2+\cdots+\left(\frac{z}{w}\right)^N\right]dw}_{\sum_{n=0}^{N}\underbrace{\left(\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w^{n+1}}\,dw\right)}_{a_n} z^n} \;+\; \text{remainder}$$

remainder $\varepsilon_N(z) \to 0$ unif on smaller disc.

Same as power series derivation

$$+\;\underbrace{\frac{1}{2\pi i}\cdot\frac{1}{z}\int_{C_r} f(w)\left[1+\left(\frac{w}{z}\right)+\cdots+\left(\frac{w}{z}\right)^N\right]dw}_{\underbrace{\sum_{n=0}^{N}\left(\frac{1}{2\pi i}\int_{C_r} f(w)\,w^n\,dw\right)\frac{1}{z^{n+1}}}_{\sum_{n=1}^{N+1}\underbrace{\left(\frac{1}{2\pi i}\int_{C_r} f(w)\,w^{n-1}\,dw\right)}_{a_{-n}}\frac{1}{z^n}}} \;+\; \text{remainder term}$$

remainder term $\varepsilon_N(z)\to 0$ unif on bigger discs

$$w^{n-1} \to \frac{1}{w^{-n+1}}$$