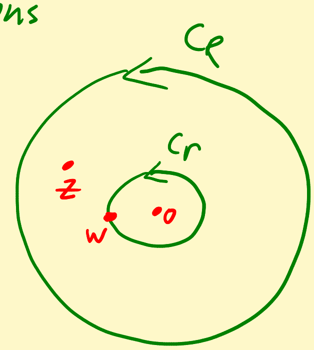
Lecture 28 Laurent expansions

$\frac{1}{2\pi i}\int_{C_R} \cdots$ power series $+$



$$-\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dz = \frac{1}{2\pi i}\int_{C_r}\frac{1}{z}f(w)\underbrace{\left[\frac{1}{1-\frac{w}{z}}\right]}_{\left[1+\left(\frac{w}{z}\right)+\cdots+\left(\frac{w}{z}\right)^N\right]+\underbrace{\frac{\left(\frac{w}{z}\right)^{N+1}}{1-\frac{w}{z}}}_{\varepsilon_N\left(\frac{w}{z}\right)}}dw$$

$$= \begin{pmatrix}\text{negative terms} \\ \sum_{n=1}^{N+1} a_{-n} z^{-n} \\ \text{in Laurent exp.}\end{pmatrix} + \varepsilon_N(z)$$

Where $\left|\varepsilon_N(z)\right| \leq \frac{1}{2\pi}\frac{1}{|z|}\left(\max_{C_r}|f|\right)\frac{\left(\frac{r}{|z|}\right)^{N+1}}{1-\frac{r}{|z|}}$

$\to 0$ as $N\to\infty$,
unif on $|z| > \tilde{r} > r$.

<u>Thm</u>: f has an isolated sing at z_0, Laurent exp: $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$

1) $a_{-n}=0\ \forall n\in\mathbb{N} \iff z_0$ removable
2) $a_{-n}\neq 0$ for only finitely many $n\in\mathbb{N} \iff z_0$ pole
3) $a_{-n}\neq 0$ for infinitely many $n\in\mathbb{N} \iff z_0$ essential

Pf: ($\Rightarrow$) If $a_{-n} = 0$ $\forall n \in \mathbb{N}$, then $f =$ power series on $D_R(z_0) - \{z_0\}$. So z_0 removable.

($\Leftarrow$) $$a_{-n} = \frac{1}{2\pi i} \int_{C_\rho} \underbrace{\frac{f(w)}{(w-z_0)^{-n+1}}}_{f(w)(w-z_0)^{n-1}} \, dw = 0 \text{ by Cauchy's}$$

$\leftarrow$ analytic in $D_\rho(z_0)$.

2) Lemma: Laurent expansions are unique.

Remark: $\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = f(z)$ means $\sum_0^{\infty}$ part and $\sum_{n=-1}^{-\infty}$ part each converge.

Why: $$a_N = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z-z_0)^{N+1}} \, dz$$

$$= \frac{1}{2\pi i} \int_{C_\rho} \left[\sum_{-\infty}^{\infty} b_n (z-z_0)^n \right] \frac{1}{(z-z_0)^{N+1}} \, dz$$

$$\underset{\substack{\uparrow \\ \text{unif conv}}}{=} \sum_{n=-\infty}^{\infty} \underbrace{\left(\frac{b_n}{2\pi i} \int_{C_\rho} (z-z_0)^{n-N-1} \, dz \right)}_{\substack{=0 \quad n \neq N \\ = 2\pi i \quad n = N}}$$

$$= b_N \checkmark$$

Pf of (2): ($\Rightarrow$) $f(z) = \underbrace{(\text{Princ part}) + \text{power series}}_{\text{Laurent expansion.}}$

so pole.

($\Leftarrow$) $f(z)$ has pole at z_0 : $f(z) = (\text{princ part}) + \text{power series}$ must be = the Laurent expansion. $a_{-n} \neq 0$ for only finitely many $n \in \mathbb{N}$.

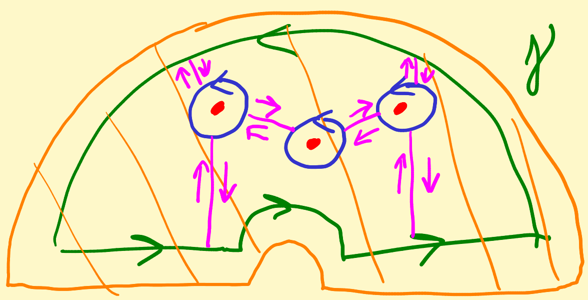
3) left over case. ✓

Interesting fact: $a_{-1} = \frac{1}{2\pi i} \int_{C_\rho} f(z)\, dz = \operatorname{Res}_{z_0} f$

Residue thm holds for small discs.

Feeling: discs $\Longrightarrow$ toy regions.

Suppose f has finitely many isolated sing inside γ.



f analytic on ▨ minus ∴

$C_{r_n}(a_n) \leftarrow$ counterclockwise

Cauchy thm on pieces : $\underbrace{\left(\int_\gamma + \sum \int_{-C_{r_n}(a_n)} \right)}_{\substack{=\text{sum of counterclockwise} \\ \text{contours with Cauchy thms.}}} f\, dz = 0$

So $\int_\gamma f\, dz = \sum \int_{C_{r_n}(a_n)} f\, dz = \sum 2\pi i \operatorname{Res}_{a_n} f$ ✓

Meaning of residue: z_0 only sing of f on $D_R(z_0)$.

f has an analytic antiderivative on $D_R(z_0) - \{z_0\}$

$\Longleftrightarrow$ $\mathrm{Res}_{z_0} f = 0$.

Pf: $(\Rightarrow)$: $f = F'$. $\quad a_{-1} = \frac{1}{2\pi i} \int_{C_r} \overbrace{F'}^{f} \, dz$

$= 0$ by Fund Thm Calc.

$(\Leftarrow)$ Assume $\int_{C_r} f \, dz = 0$.

Lemma f has an analytic antiderivative on a domain Ω if and only if $\int_\gamma f \, dz = 0$ for every closed γ in Ω.

Pf: $(\Rightarrow)$: Fund thm Calc. ✓

$(\Leftarrow)$: Hmmm. On convex $\quad F(z) = \int_{L_a^z} f(w) \, dw$

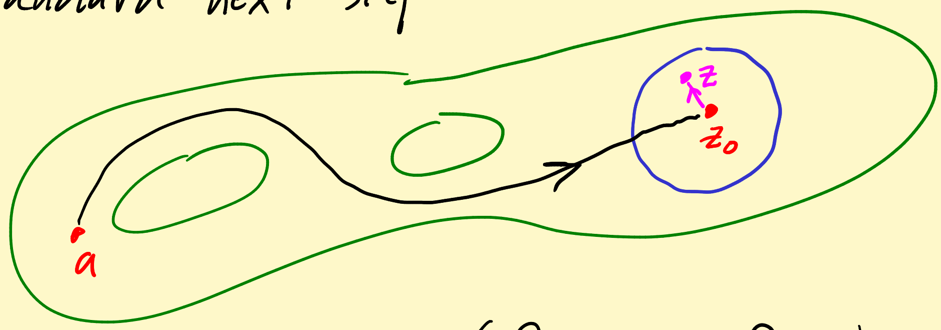
Try: Fix $a \in \Omega$. Let γ_a^z denote a curve from a to z in Ω.

"Define" $\quad F(z) = \int_{\gamma_a^z} f(w) \, dw$

F is well defined: $\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z} \right) f \, dw = \int_\Gamma f \, dw = 0$

because $\gamma_a^z \cup (-\tilde{\gamma}_a^z) = \Gamma$ is a closed contour.

Standard next step



$$F(z) = \left(\int_{\gamma_a^{z_0}} + \int_{L_{z_0}^z} \right) f \, dw$$

$$= (\text{const}) + \underbrace{\int_{L_{z_0}^z} f \, dw}_{\text{anti deriv for } f \text{ on Disc}}$$

Finally, suppose γ is a closed curve in $D_R(z_0) - \{z_0\}$.

Suppose $a_{-1} = \operatorname{Res}_{z_0} f = 0$.

$$\int_\gamma f \, dz = \int_\gamma \underbrace{\sum_{n \neq -1} a_n (z - z_0)^n}_{\substack{\text{converges} \\ \text{unif on} \\ A(\rho_1, \rho_2)}} \, dz$$

$\rho_1 = \min |z(t) - z_0|$ $\qquad \gamma : z(t)$

$\rho_2 = \max |z(t) - z_0|$

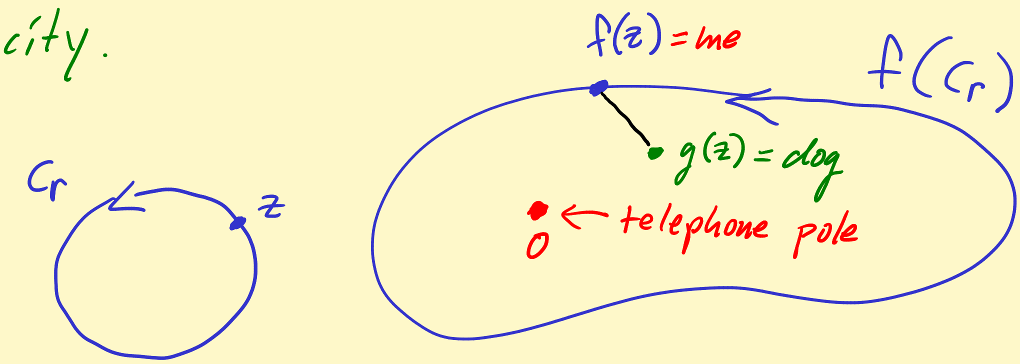
$$= \sum_{n \neq -1} a_n \underbrace{\int (z - z_0)^n \, dz}_{=0}$$

Next time: Rouché's thm: Suppose $f(z)$ and $g(z)$ are analytic on $D_R(a)$ and $0<r<R$. If

$$|f(z)-g(z)| < |f(z)| \text{ on } C_r(a),$$

then f and g have same # zeroes inside $C_r(a)$ counted with multiplicity.

Feeling:



leash length < dist (me, pole)

Arg princ: $\left(\begin{array}{c}\text{\# zeroes of} \\ \text{in } C_r\end{array}\right)$ = # times I go around pole

= same # times dog does, too.