Lecture 29 Rouché's thm & Hurwicz thms

HWK 7 posted on home page. Due Fri, Nov 17

<u>Rouché's theorem</u> for a disc (or "Toy region). Suppose f and g are analytic on $D_R(a)$ and suppose $0<r<R$ and

$$(*) \quad |f(z)-g(z)| < |f(z)| \text{ on } C_r(a), \text{ then}$$

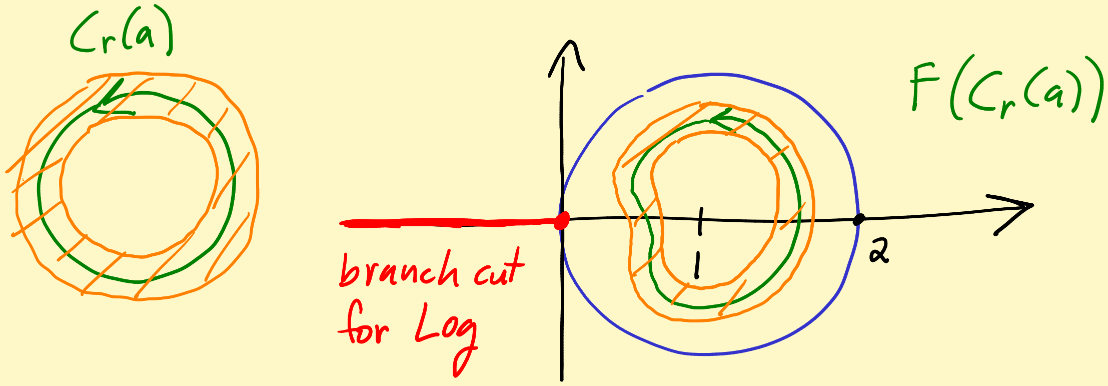
f and g have the same # of zeroes inside $C_r(a)$ (counted with multiplicity).

<u>Pf</u>: $(*) \Rightarrow f$ has no zeroes on $C_r(a)$

$(*) \Rightarrow g$ has no zeroes on $C_r(a)$ too.

<u>Step 1</u> Divide $(*)$ by $|f(z)|$:

$$\Big|1 - \underbrace{\frac{g(z)}{f(z)}}_{F(z)}\Big| < 1 \quad \text{on } C_r(a)$$



$$\frac{1}{2\pi i}\int_{C_r(a)} \frac{F'}{F}\,dz = \frac{1}{2\pi i}\int_{C_r(a)} \frac{d}{dz}\,\mathrm{Log}\,F(z)\,dz = 0$$

<u>Fact</u>: $\frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f}$ "Logarithmic diff'tion"

$$\underbrace{\frac{1}{2\pi i}\int_{C_r(a)} \frac{g'}{g}\,dz}_{\#\text{ zeroes } g} - \underbrace{\frac{1}{2\pi i}\int_{C_r(a)} \frac{f'}{f}\,dz}_{\#\text{ zeroes } f} = 0 \quad \checkmark$$

Remark Same pf works for Toy regions and case where f and g have finitely many poles inside curve.

$$[\#\text{ zeroes} - \#\text{ poles}] \text{ of } f = \text{same } \# \text{ for } g.$$

Fun thing: Rouché's $\Rightarrow$ Fund Thm Alg

$$f(z) = a_N z^N \qquad g(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_0$$

$$|\underbrace{f(z) - g(z)}_{\text{poly deg } N-1}| \overset{?}{<} |\underbrace{f(z)}_{\text{deg } N}| \quad \text{on } C_R(0)$$

Right hand side of basic poly est $\Rightarrow$ result.

EX: How many zeroes does $\underbrace{z^5 + 3z^4 + \underbrace{8z^3}_{f(z) \text{ big one}} + z^2 + z + 1}_{g(z)}$ have in unit disc?

$$|f(z) - g(z)| = |z^5 + 3z^4 + z^2 + z + 1| \underset{\text{on } |z|=1}{\leq} \underbrace{1 + 3 + 1 + 1 + 1}_{7} < 8 \underbrace{|z^3|}_{1}$$

g has 3 zeroes.

<u>Hurwicz thm #1</u> Suppose f_n is a seq of analytic fcns on a domain Ω and $f_n \rightrightarrows f$ there. (unif conv on compacts)

[Know f analytic too via Morera's.]

If all the f_n are non-vanishing on Ω, then either

1) $f \equiv 0$ on Ω, <u>or</u>

2) f is non-vanishing on Ω too.

Way false $\mathbb{R} \to \mathbb{R}$: $h_n(t) = t^2 + \frac{1}{n}$

<u>EX</u>: $f_n(z) = z^n$ on $\Omega = D_1(0) - \{0\}$. $f(z) \equiv 0$.

<u>Pf of Hurwicz</u> Suppose $f(z_0) = 0$. If $f \not\equiv 0$, then z_0 is an <u>isolated</u> zero of f. So $\exists\ \overline{D_\varepsilon(z_0)} \subset \Omega$ so that z_0 is the <u>only</u> zero of f on $\overline{D_\varepsilon(z_0)}$.

Let $m = \min\{|f(z)| : z \in C_\varepsilon(z_0)\} > 0$

Want $\quad |f_n(z) - f(z)| < m < |f(z)|$ on $C_\varepsilon(z_0)$

ε in unif conv

compact set.

So $\exists\ N$ such that f_n and f have same # zeroes in $C_\varepsilon(z_0)$. $\left.\begin{array}{l} f \text{ has } \underline{\text{at least one}} \\ f_n \text{ have none} \end{array}\right\} \lightning$

So $f \equiv 0$ ✓

Hurwicz #2 Ω, $f_n \rightrightarrows f$ like #1.

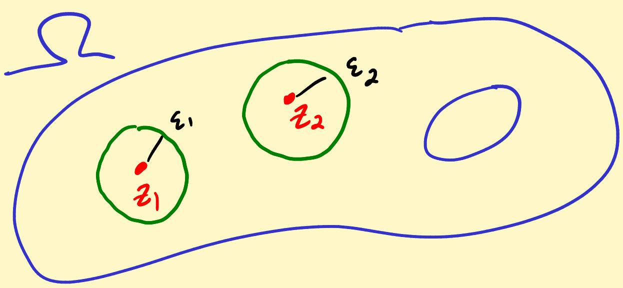
Suppose f_n are 1-1 on Ω. Then either

1) $f \equiv$ const on Ω, or
2) f is 1-1 too.

Pf Redo pf of H #1. Suppose $f(z_1) = f(z_2) = w_0$

$z_1, z_2 \in \Omega$, $z_1 \neq z_2$. [f is not 1-1.]

$f(z) - w_0$ has zeroes at z_1 and z_2. If $f \not\equiv w_0$, then z_1 and z_2 are isolated zeroes.



Shrink $\varepsilon_1, \varepsilon_2$ so discs are disjoint, one zero, etc.

Rouché argument => for

$n > \text{Max}(N_1, N_2)$:

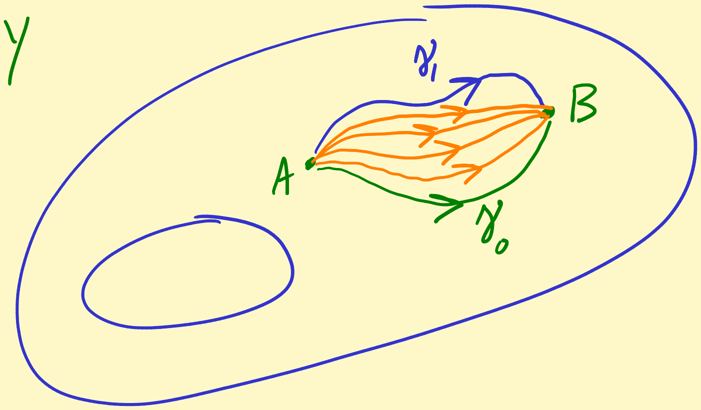
$$\begin{pmatrix} \#\text{ zeroes of } f_n(z) - w_0 \\ \text{inside } C_\varepsilon(z_j) \end{pmatrix} = \underbrace{\begin{pmatrix} \#\text{ zeroes of } f(z) - w_0 \\ \text{inside } C_\varepsilon(z_j) \end{pmatrix}}_{\geq 1}$$

for $j = 1, 2$.

So $\exists\ \zeta_j$ inside $C_\varepsilon(z_j)$ with $f_n(\zeta_j) = w_0$

f_n not 1-1 ↯.

Homotopy



$\gamma_j : z_j(t), \; 0 \le t \le 1$

$\gamma_0 \sim \gamma_1$: "γ_0 is homotopic to γ_1 in Ω" means

there is a continuous $H : [0,1] \times [0,1] \longrightarrow \Omega$

such that
$$H(0,t) = z_0(t) \qquad 0 \le t \le 1$$
$$H(1,t) = z_1(t) \qquad 0 \le t \le 1$$

$$\begin{cases} H(s,0) = A \\ H(s,1) = B \end{cases} \quad \text{for } 0 \le s \le 1$$

Think: $\gamma_s : z_s(t) = H(s,t) \leftarrow$ orange curves.