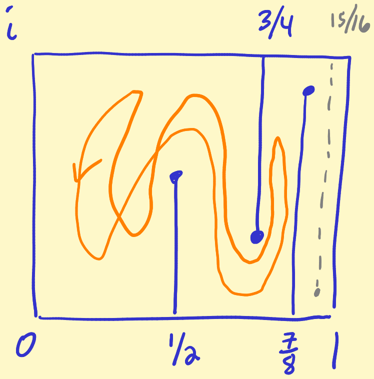
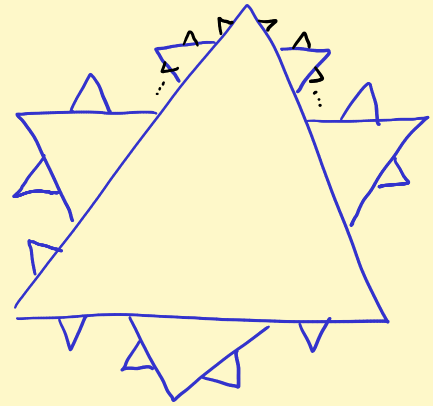
Lecture 31 Simply connected domains in the plane



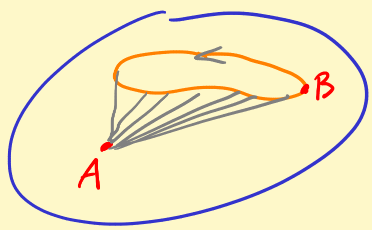


Sierpinski snowflake

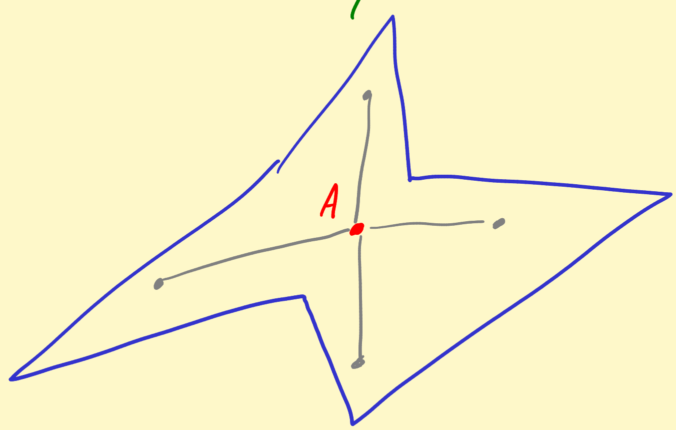
$A \leq \frac{L^2}{4\pi}$ Isoparametric ineq

$\hookrightarrow \infty$ as points get added

Convex and star shaped domains are simply connected



"Tornedo homotopy"



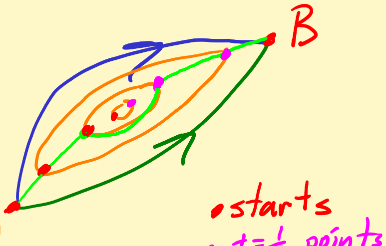
Topology facts

1) Ahlfors equiv def^n of s.c. domain Ω : $\hat{\mathbb{C}} - \Omega$ is connected.

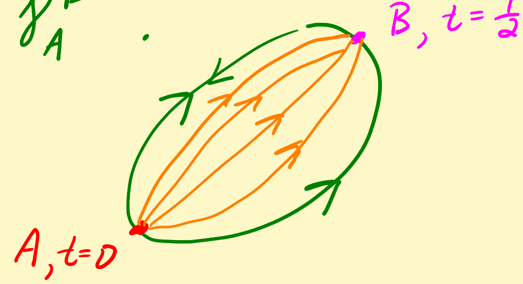
2) Ω is simply connected $\Longleftrightarrow$ Given any two points $A, B \in \Omega$ and curves $\gamma_A^B, \tilde{\gamma}_A^B$.

$\gamma_A^B \sim \tilde{\gamma}_A^B$.

$(\Rightarrow)$

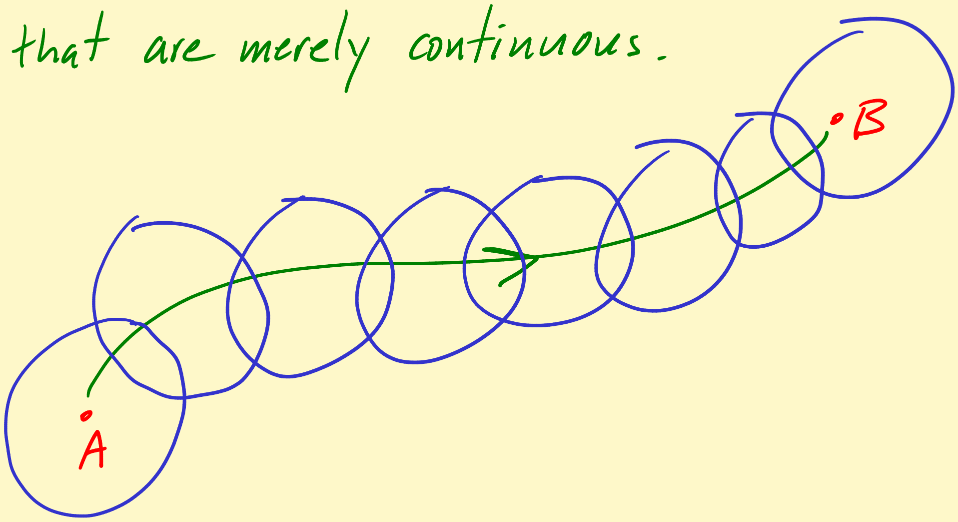


$(\Leftarrow)$



push orange curves over to other curve, suck in like spaghetti

Interesting consequence: f analytic, $\int_\gamma f\,dz$ can be defined for γ that are merely continuous.



Diameters of discs are chord approx to curve

Get s.c. "caterpillar"! $\int_\gamma$ is path indep!

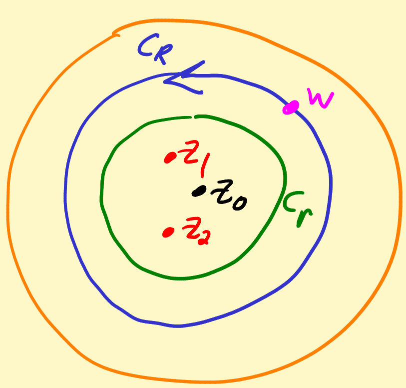
Arzela-Ascoli: Baby Rudin p. 158
Ahlfors p. 219–227
Stein p. 225
Taylor p. 255

Big lemma Uniformly bounded family $\mathcal{F}$ of analytic fcns is uniformly equicontinuous on compact sets.

Pf: Cauchy integral!

$f \in \mathcal{F}$

$\max_{C_R} |f| < M$ ← indep of $f \in \mathcal{F}$



$D \subset \Omega$

$$\left| f(z_1) - f(z_2) \right| = \left| \frac{1}{2\pi i} \int_{C_R} f(w) \underbrace{\left[\frac{1}{w-z_1} - \frac{1}{w-z_2} \right]}_{\frac{z_1 - z_2}{(w-z_1)(w-z_2)}} dw \right|$$

$$\leq \frac{1}{2\pi} \underbrace{\left(\max_{C_R} |f| \right)}_{\leq M} |z_1 - z_2| \frac{1}{(R-r)^2} 2\pi R$$

$$< \underbrace{(\text{const})}_{\text{indep of } f} |z_1 - z_2|$$

Given compact $K \subset \Omega$, for each pt $z_0 \in K$, get $D_r(z_0)$ as in pic. Get cover. Take a finite subcover. Given ε, get finitely many δ's. Take min δ to get a δ that works on K for all $f \in \mathcal{F}$.

<u>Montel's thm</u> Suppose f_n are analytic fcns on a domain are uniformly bounded on compact subsets of Ω. Then there is a subseq f_{n_k} that converges unif on compacts to f (which is analytic).

$$\int_{-\infty}^{\infty} \frac{z^2}{(z^2+a^2)^3}\,dz = 2\pi i \operatorname{Res}_{ia} \frac{z^2}{(z^2+a^2)^3}$$

$$\frac{z^2}{\left[(z-ia)(z+ia)\right]^3}$$

$\cdot ia$

$\cdot -ia$

$$\frac{1}{(z-ia)^3}\left[\underbrace{\frac{z^2}{(z+ia)^3}}_{H(z)=A_0+A_1(z-ia)+A_2(z-ia)^2+\cdots}\right]$$

$$= \frac{A_0}{(z-ia)^3} + \frac{A_1}{(z-ia)^2} + \frac{A_2}{z-ia} + \cdots$$

$$A_2 = H''(ia)/2! = \operatorname{Res}$$

Fact: If $\operatorname{Deg} P(z) \geq 2$, $\displaystyle\int_{C_R} \frac{1}{P(z)}\,dz = 0$

if C_R contains all zeroes of P.

Pf: See $\displaystyle\int_{C_R} \to 0$ as $R \to \infty$. Homotopy shows $\displaystyle\int_{C_R}$ are constant. Constant must be zero.