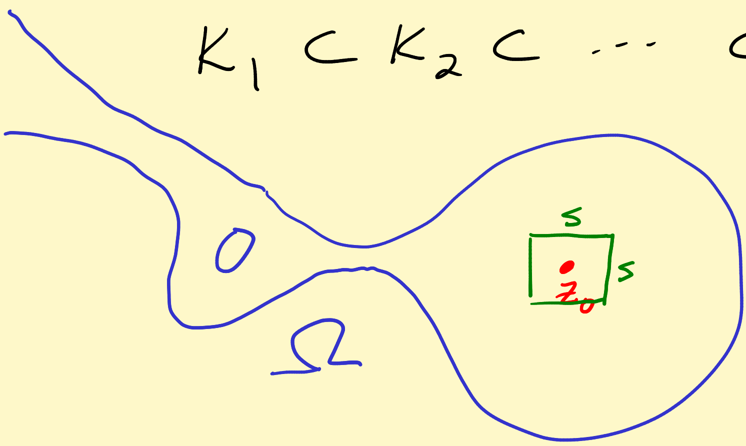
Lecture 32 Proof of Montel's theorem, prep for pf of R.Map.Thm.

Montel's thm f_n a seq of analytic fcns on a domain Ω that is uniformly bounded on compact subsets of Ω. Then there exists a subseq $f_{n_k} \twoheadrightarrow$ analytic f.

Pf **Step 1** "Exhaust" Ω by compact sets

$$K_1 \subset K_2 \subset \cdots \subset \Omega, \quad \Omega = \bigcup K_j$$



$K_1 = \left(\text{Union of closed squares of side } s \subset \Omega\right) \cap D_{10s}(z_0)$

K_2 = Same, but use $\frac{s}{2}$ in place of s and $D_{100s}(z_0)$ ...

K_3 = " $\frac{s}{2^2}$ . . . . $D_{100s}(z_0)$ ...

⋮

Fancier version: $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega$ domains,

$$K_j = \overline{\Omega_j}. \qquad K_j \subset \mathring{K}_{j+1}$$

Step 2: $z \in \mathbb{C}$: $\operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}$ is a countable dense subset of $\mathbb{C}$.

Fix K_j. Let $\{z_n\}_{n=1}^{\infty}$ be a countable dense subset of K_j from above dense subset of $\mathbb{C}$.

Step 3 $f_1(z_1), f_2(z_1), f_3(z_1), \ldots$ is an bnd seq

So it has a conv subseq:

$$f_{1,1}(z_1), f_{1,2}(z_1), f_{1,3}(z_1), \ldots \longrightarrow \text{conv.}$$

Now $f_{1,1}(z_2), f_{1,2}(z_2), f_{1,3}(z_2), \ldots$ is a bnd seq

It has a conv subseq

$$f_{2,1}(z_2), f_{2,2}(z_2), f_{2,3}(z_2), \ldots \longrightarrow \text{conv}$$

Now $f_{2,1}(z_3), f_{2,2}(z_3), f_{2,3}(z_3), \ldots$ is a bnd seq

It has a conv subseq

$$f_{3,1}(z_3), f_{3,2}(z_3), f_{3,3}(z_3), \ldots \longrightarrow \text{conv.}$$

etc.

Aha! $\underbrace{f_{1,1}}_{F_1}, \underbrace{f_{2,2}}_{F_2}, \underbrace{f_{3,3}}_{F_3}, \ldots$ is a subseq that converges at every z_n!

Step 4 F_n are uniformly Cauchy on K_j. Hence they converge uniformly on K_j. Let $\varepsilon > 0$.

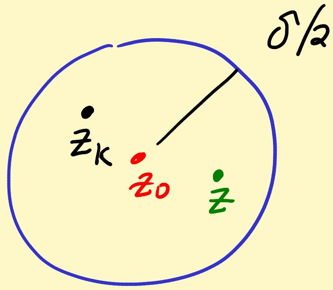
$$\Big|\underset{\substack{\uparrow \\ z \in K_j}}{F_n(z)} - F_m(z)\Big| = \Big|\underbrace{F_n(z) - F_n(z_K)}_{|\ \ | < \frac{\varepsilon}{3} \text{ pick } z_K \text{ close to } z \text{ and use equi cont}} + \underbrace{F_n(z_K) - F_m(z_K)}_{|\ | < \frac{\varepsilon}{3} \text{ because } F_n(z_K) \text{ conv.}} + \underbrace{F_m(z_K) - F_m(z)}_{|\ \ | < \frac{\varepsilon}{3} \text{ same reason as term \#1.}}\Big|$$

Can get N such that $n, m > N \implies |F_n(z) - F_m(z)| < \varepsilon$ on K_j.

Need an N that works on all of K.

How: Pick $z_0 \in K_j$. Get $\delta > 0$ for equi cont for $\frac{\varepsilon}{3}$ on K_j.

Hmmm. $D_{\delta/2}(z_0)$



$|z - z_K| < \delta$

So outside terms $|\quad| < \frac{\varepsilon}{3}$.

Since $F_n(z_K)$ conv, it is unif Cauchy, so $\exists N$ such that middle term $|\quad| < \frac{\varepsilon}{3}$ when $n, m > N$.

Cover K_j by such discs $D_{\delta/2}(z_0)$. Take finite subcover. $N = \text{Max } N\text{'s}$ for discs.

Step 5 Get subseq $F_{1,1}, F_{1,2}, F_{1,3}, \ldots$ conv unif on K_1

take subseq $F_{2,1}, F_{2,2}, F_{2,3}, \ldots$ conv unif on K_2

take subseq $F_{3,1}, F_{3,2}, F_{3,3}, \ldots$ conv unif on K_3

etc. $\vdots$

$F_{n,n}$ is the desired subseq that converges unif on each K_j!

Step 6 Given compact $K \subset \Omega$, $\exists$ K_j that we constructed with $K \subset K_j$. So subseq conv unif on K.

Technical point: We want to use Hurwicz #2 about limits of 1-1 analytic fcns having 1-1 limits if not const to get Riemann map = limit of 1-1 fcns.

<u>Case</u>: $\Omega = \mathbb{C}$. There is no $f : \Omega \to D_1(0)$ analytic and 1-1. [f bdd entire $\Rightarrow f \equiv$ const.]

<u>Case</u> Ω <u>bounded</u> domain: $M = \sup\{|z| : z \in \Omega\}$.

$$f(z) = \frac{z}{M+1}. \quad f \text{ is analytic} : \Omega \xrightarrow[1\text{-}1]{} D_1(0).$$

<u>Case</u> $\Omega \neq \mathbb{C}$ simply connected.

$$\exists \ z_0 \in \mathbb{C} - \Omega.$$

Then $z - z_0$ is <u>non-vanishing</u> on s.c. Ω.

So $\exists$ analytic $g(z)$ on Ω such that

$$g(z)^2 = z - z_0 .$$

<u>Claim</u>: $g(z)$ is 1-1 on Ω.

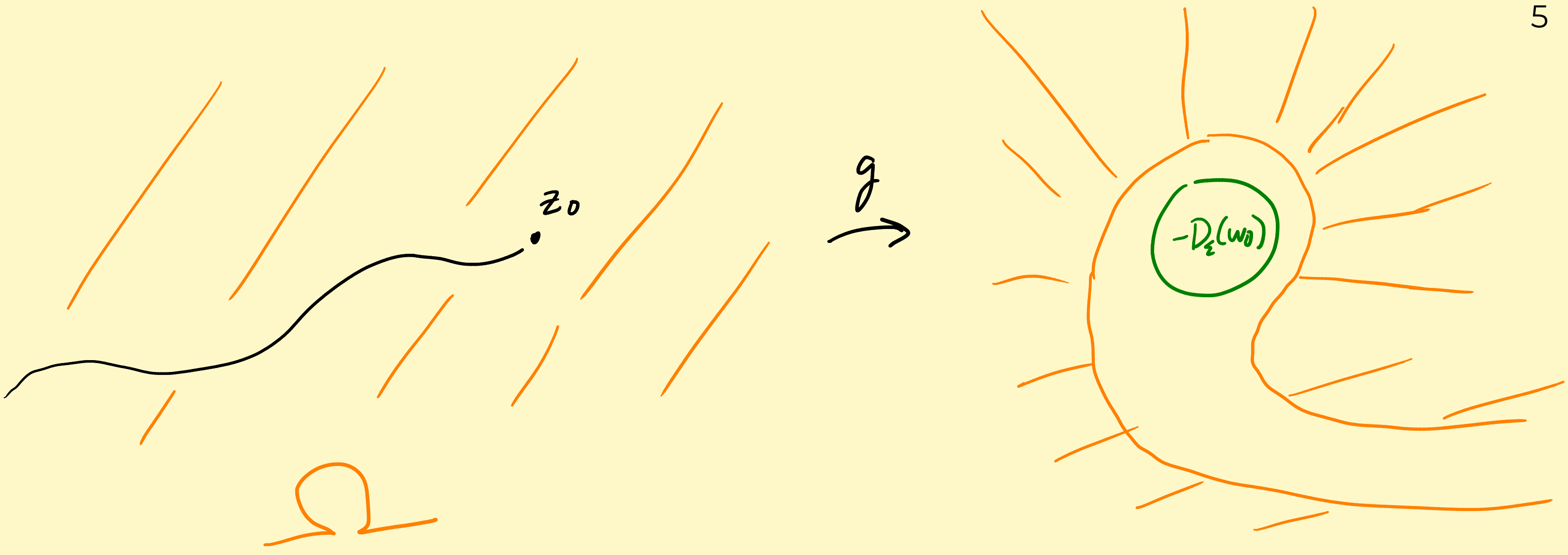
$$\begin{cases} g(z_1) = g(z_2) \\ g(z_1)^2 = g(z_2)^2 \\ z_1 - z_0 = z_2 - z_0 \implies z_1 = z_2 \ \checkmark \end{cases}$$

<u>Claim</u>: g misses a disc on Ω.

<u>Why</u>: g not const. $\checkmark$

Open mapping thm $\Rightarrow \exists \ D_\varepsilon(w_0) \subset g(\Omega)$.

Note: g 1-1 on $\Omega \Rightarrow -D_\varepsilon(w_0) = \{-z : z \in D_\varepsilon(w_0)\}$ is <u>not</u> $\subset g(\Omega)$.



Get 1-1 analytic fcn $\longrightarrow D_1(0)$ ✓