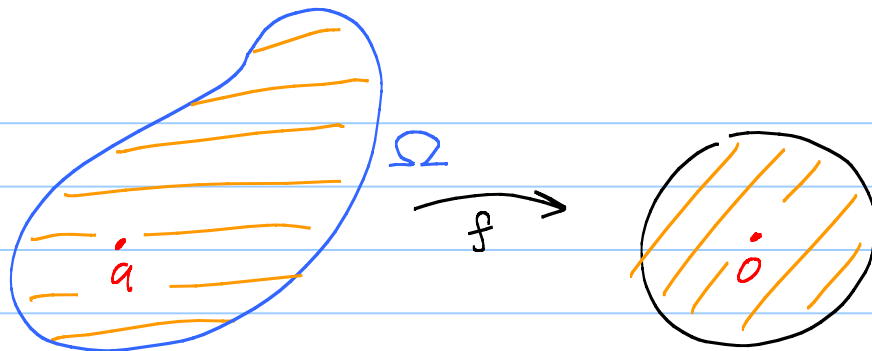
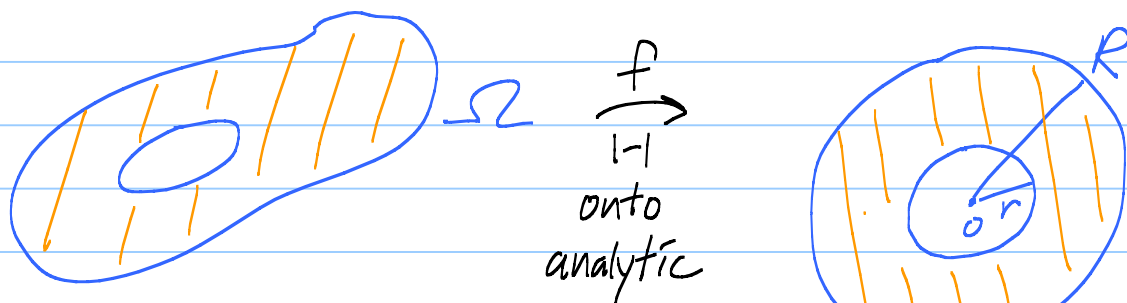


Riemann map

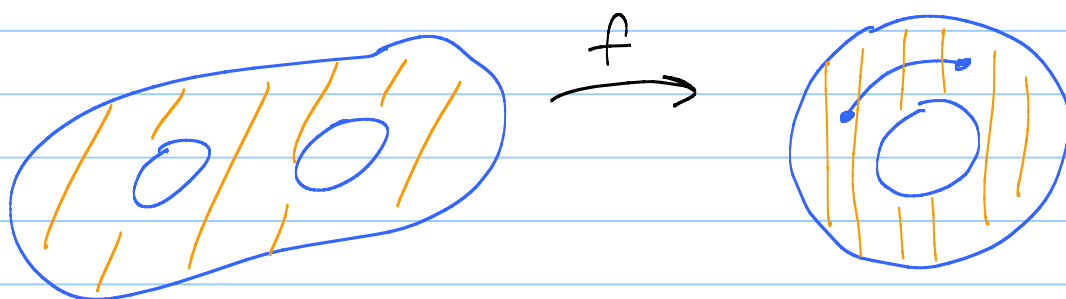
The Riemann map $|f'_a(a)| = \text{Max} \{ |h'(a)| : h: \Omega \xrightarrow{\text{analytic}} D_1(o) \}$
 Unique: $f'_a(a) \in \mathbb{R}^+$ ← f_a solves this extremal problem ↑ not assuming 1-1
 Always: $f_a(a) = 0$.

What about domains with holes?

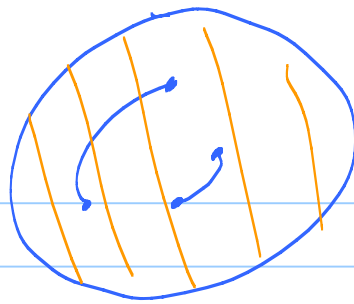
$\frac{r}{R}$ determines a "mapping class". $\frac{r}{R}$ = the "modulus" of Ω

$$\text{Aut}(\text{Annulus}) = \{ \lambda z : |\lambda| = 1 \} \cup \{ \text{inversion} \sim \frac{1}{z} \}$$

not transitive!

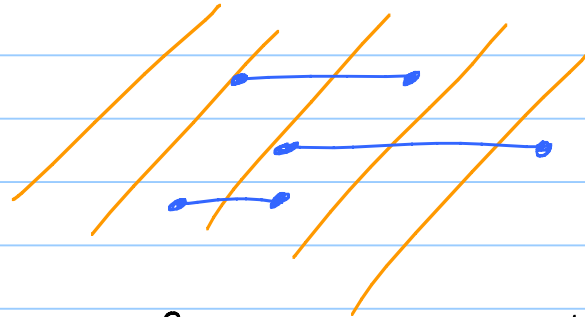


f_2



Disc minus
circular
slits

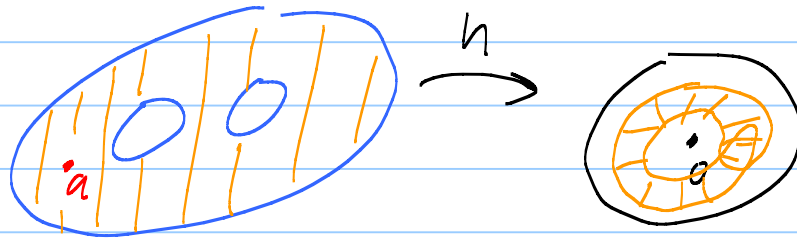
f_3



C-horiz slits.

3-connected domains and above: finitely many automorphisms, perhaps (most likely) none.

Ahlfors map:



Extremal map $f_a = \{f'_a(a) = \text{Max} \{ |h'(a)| : h: \Omega \rightarrow D(a) \} \}$
analytic

Unique: $f'_a(a) \in \mathbb{R}^+$

f_a is the Ahlfors map. Like a Riemann map
for n -connected domains. ($n > 1$).

Onto $D_1(0)$ ✓

Analytic ✓

Not 1-1. It is n -to-one, counting multiplicity.

Maps each boundary 1-1 onto unit circle.

3

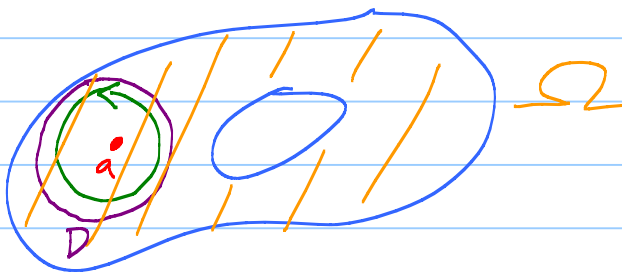
Harmonic fcn on a domain Ω is a C^2 -smooth real valued fcn such that $\Delta u \equiv 0$.

Fact: $\Leftrightarrow$ u is locally the real part of an analytic fcn.

Remark: Can get global harmonic conjugates when Ω is simply connected.

$[\ln|z|, \text{ no harm conj on } \mathbb{C} \setminus \{0\}]$

Important property: The averaging property



$$(*) \quad u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \underbrace{r d\theta}_{ds}$$

Why: Get $f = u + iv$ on D , analytic.

Cauchy integral formula at a :

$$\underbrace{f(a)}_{u+iv} = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{f(a + re^{i\theta})}_{u+iv} d\theta$$

Take real part to get $(*)$.

Thm: Ave prop $\Rightarrow$ Max principle.

Why: Suppose u has a local max at $z_0 \in \Omega$.

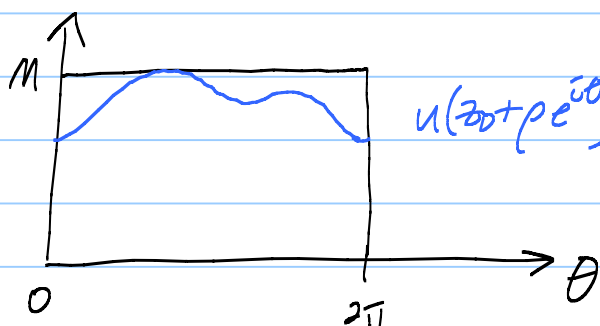
$\exists D_\varepsilon(z_0) \subset \Omega$ such that $u(z) \leq u(z_0) = M$
on $D_\varepsilon(z_0)$.

Ave prop

$$\underbrace{u(z_0)}_M = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \underbrace{\rho e^{i\theta}}_{0 < \rho < r} d\theta$$

Thing inside $\int$ is $\leq M$ and continuous.

Calculus lemma, $\Rightarrow u(z_0 + \rho e^{i\theta}) \equiv M, \forall \theta, 0 < \rho < r$.



$u(z_0 + \rho e^{i\theta}) \leftarrow$ picture can't happen!
 $u \equiv M$.

Conclude that $u \equiv M$ on $D_\varepsilon(z_0)$.

Aha! $F = u_x - i u_y$ is analytic on Ω

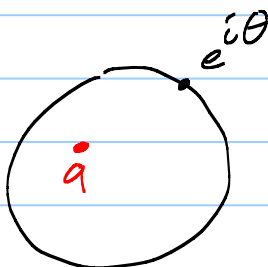
$u \equiv M$ on $D_\varepsilon \Rightarrow F \equiv 0$ on $D_\varepsilon \Rightarrow F \equiv 0$ on Ω ,

$\Rightarrow \nabla u \equiv 0$ on $\Omega \Rightarrow u \equiv \text{const}$ on Ω .
 $\uparrow = M. \checkmark$

Want a "Cauchy integral formula" for harmonic fns.

See "Something about Poisson and Dirichlet." Find a link below AFTER MATH on home page.

Poisson integral formula:



$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|a|^2}{|a-e^{i\theta}|^2} u(e^{i\theta}) d\theta$$

Simple Pf, easy to understand.

Formula true at $a=0$. Ave prop. ✓

Hmmm. $\varphi_a(z) = \frac{z+a}{1+\bar{a}z}$: $D_1(0) \xrightarrow{1-1} \text{onto} \varphi_a(0) = a$.

Fact: harmonic fns composed with analytic fns are harmonic.

Aha! Ave prop:

$$u(\underbrace{\varphi_a(0)}_a) = \frac{1}{2\pi} \int_0^{2\pi} u(\underbrace{\varphi_a(e^{i\theta})}_{e^{i\psi}}) d\theta$$

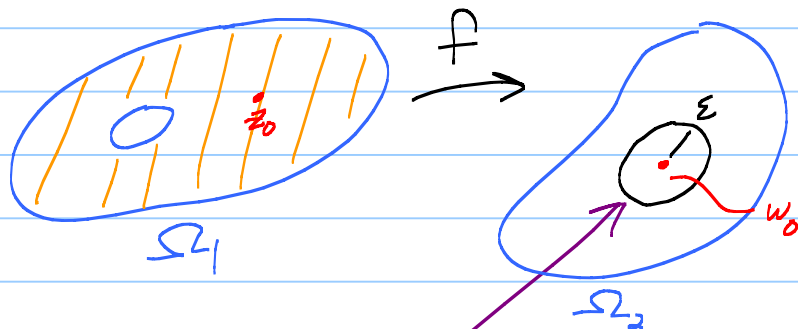
$$e^{i\psi} = \frac{e^{i\theta} + a}{1 + \bar{a}e^{i\theta}}$$

Calculus: $d\theta = \frac{1-|a|^2}{|a-e^{i\psi}|^2} d\psi$

Get $u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\psi}) \frac{1-|a|^2}{|a-e^{i\psi}|^2} d\psi$ ✓

Fact $f: \Omega_1 \rightarrow \Omega_2$ analytic, u harmonic on Ω_2 .
Then $u \circ f$ is harmonic on Ω_1 .

Why;



$$f(z_0) = w_0$$

Ω_2 open: $\exists D_\varepsilon(w_0) \subset \Omega_2$
 f analytic $\Rightarrow f$ cont.
 So $\exists \delta > 0$ so that
 $D_\delta(z_0) \subset \Omega_1$ and
 $f(D_\delta(z_0)) \subset D_\varepsilon(w_0)$.

$u = \operatorname{Re} g$, g analytic
on $D_\varepsilon(w_0)$.

Finally $u \circ f = \operatorname{Re}[g \circ f]$ on $D_\delta(z_0)$.

Real part of analytic fcn.
 So $u \circ f$ is harmonic on $D_\delta(z_0)$.