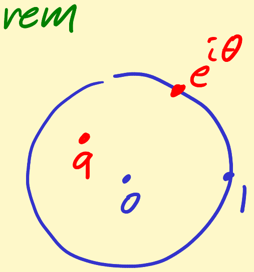
Lecture 35 The Dirichlet problem, Schwarz's theorem

Poisson kernel : $P(a,\theta) = \frac{1}{2\pi} \frac{1-|a|^2}{|e^{i\theta}-a|^2}$



Assume u harmonic on $D_R(0)$, $R>1$. Then

$$u(a) = \int_0^{2\pi} P(a,\theta)\, u(e^{i\theta})\, d\theta \qquad (*)$$

<u>Remark</u> $(*)$ holds if just u is continuous on $\overline{D_1(0)}$ and harmonic inside.

<u>Why</u> $u_r(z) = u(rz)$, $0<r<1$, is harmonic on $D_R(0)$ where $R = \frac{1}{r} > 1$. $(*)$ holds for $u_r(z)$. Uniform continuity of u on compact $\overline{D_1(0)}$ $\Longrightarrow$ $u_r(z) \to u(z)$ <u>uniformly</u> on $\overline{D_1(0)}$. Let $r \nearrow 1$ to see $(*)$ holds for u.

<u>Remark</u> Same trick shows Cauchy integral formula holds for analytic fcns on discs, continuous up to bndry.

<u>Dirichlet problem</u> Given continuous $U(\theta)$ on $[0,2\pi]$ with $U(0)=U(2\pi)$, is there a fcn u continuous on $\overline{D_1(0)}$ and <u>haromonic</u> on $D_1(0)$ such that $u(e^{i\theta}) = U(\theta)$?

Guess: $u(a) = \int_0^{2\pi} P(a,\theta)\, U(\theta)\, d\theta$

Schwarz's theorem: Guess works!

Solves heat problem: Know temp of metal plate on boundary. Let it reach "steady state."

Temp fcn u is harmonic and solves D. Prob.

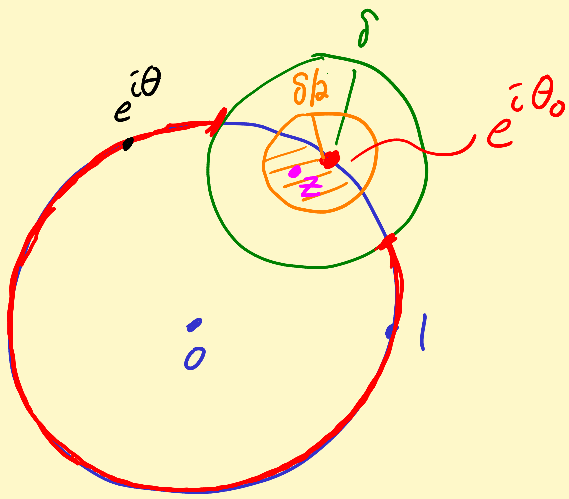
Heat eqn: $\Delta u = \frac{\partial u}{\partial t} \to 0$ steady state.

4 Key properties $P(a,\theta)$

1) $P(a,\theta) > 0$ ✓ formula

2) $\int_0^{2\pi} P(a,\theta)\, d\theta = 1$ ✓ formula: $u(z) \equiv 1$.

3)



$I_\delta = \{\theta : e^{i\theta} \notin D_\delta(e^{i\theta_0})\}$

$P(z,\theta) \le \frac{4}{\pi \delta^2}(1-|z|)$

$\to 0$ uniformly on I_δ

when $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$

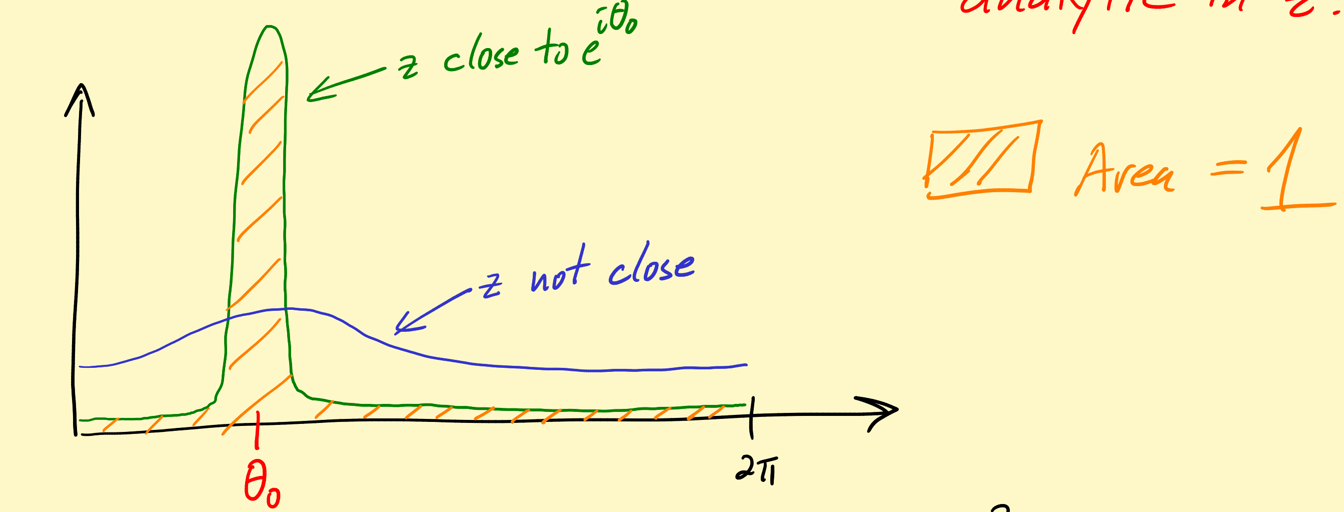
and $z \to e^{i\theta_0}$.

(3) easy: $\frac{1}{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|^2} \le \frac{1}{2\pi}\frac{1-|z|^2}{\left(\frac{\delta}{2}\right)^2} = \frac{1}{2\pi}\frac{(1-|z|)(1+|z|)}{\delta^2/4} \le \frac{1-|z|}{\pi\delta^2/4}$ ✓

$[1+|z| \le 1+1 = 2]$

4) $P(z,\theta)$ is harmonic $z \in D_1(0)$ because

$$P(z,\theta) = \frac{1}{2\pi}\,\mathrm{Re}\underbrace{\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]}_{\text{analytic in } z.}$$



Pf of Schwarz

Define $u(z) = \int_0^{2\pi} P(z,\theta)\,U(\theta)\,d\theta$

$$= \frac{1}{2\pi}\,\mathrm{Re}\underbrace{\int_0^{2\pi}\frac{e^{i\theta}-z}{e^{i\theta}+z}\,U(\theta)\,d\theta}_{\text{analytic: HWK 2: prob 1.}}$$

So u is harmonic on $D_1(0)$.

Step 2: Show $\lim_{z \to e^{i\theta_0}} u(z) = U(\theta_0)$

$$u(z) - U(\theta_0) = \int_0^{2\pi} P(z,\theta)\left[U(\theta) - U(\theta_0)\right]d\theta$$

because $\int_0^{2\pi} P(z,\theta)\,d\theta = 1$

Let $\varepsilon > 0$. $\exists\, \delta > 0$ such that $|U(\theta) - U(\theta_0)| < \frac{\varepsilon}{2}$ when $e^{i\theta} \in D_\delta(e^{i\theta_0})$.

$$I = \underbrace{\int_{I_\delta} \underbrace{P(z,\theta)}_{\text{small}} \big(U(\theta) - U(\theta_0)\big)\,d\theta}_{I_1} + \underbrace{\int_{\underbrace{[0,2\pi]-I_\delta}_{\text{small interval about } \theta_0}} P(z,\theta) \big(\underbrace{U(\theta) - U(\theta_0)}_{|\ \ | < \frac{\varepsilon}{2} \text{ small}}\big)\,d\theta}_{I_2}$$

Suppose $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$.

$$|I_1| \le \frac{4}{\pi \delta^2}(1 - |z|)\, 2M \underbrace{\mathrm{Length}(I_\delta)}_{< 2\pi} \qquad M = \max_{[0,2\pi]} |U|$$

$\longrightarrow 0$ as $z \to e^{i\theta_0}$ inside $D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$.

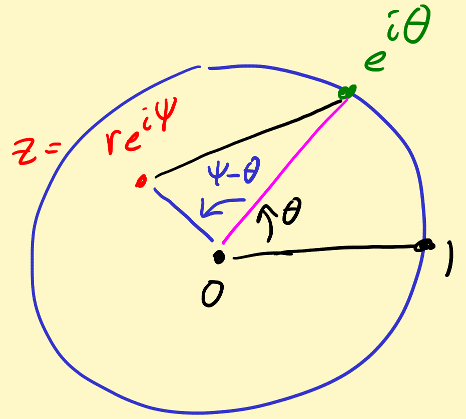
Can make $< \frac{\varepsilon}{2}$.

$$|I_2| \le \int_{[0,2\pi]-I_\delta} \underset{P>0}{P(z,\theta)} \underbrace{|U(\theta) - U(\theta_0)|}_{\le \frac{\varepsilon}{2}}\,d\theta$$

$$\le \frac{\varepsilon}{2} \int_{[0,2\pi]-I_\delta} P(z,\theta)\,d\theta \underset{\text{because } >0}{<} \frac{\varepsilon}{2} \underbrace{\int_0^{2\pi} P(z,\theta)\,d\theta}_{=1} = \frac{\varepsilon}{2}$$

Done!

Other formulas



Law of Cosines

$$|re^{i\psi} - e^{i\theta}|^2 = 1^2 + r^2 - 2r\cos(\psi - \theta)$$

$$u(re^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\psi-\theta)}\, U(\theta)\, d\theta$$

$D_R(z_0)$ version: $f(z) = \dfrac{z - z_0}{R}$

$$u(z_0 + re^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\psi-\theta)}\, u(z_0 + Re^{i\theta})\, d\theta$$

See ave prop: $u(z_0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta.$

Explosion of theorems!

Monday: A continuous real valued fcn u on a domain Ω that satisfies ave prop on discs $\overline{D_R(z_0)} \subset \Omega$ is <u>harmonic</u>!