Lecture 37 Weak ave property and reflection princ for harmonic fcns

"Something about Poisson and Dirichlet" on home page

Thm If a seq u_n of harmonic fcns converges uniformly on compact subsets of a domain Ω to u, then u is harmonic. Furthermore, $\frac{\partial^{n+m}}{\partial x^n \partial y^m} u_k \rightrightarrows \frac{\partial^{n+m}}{\partial x^n \partial y^m} u$ on Ω.

Pf: MA 504: Unif limits of cont fcns are cont. So u is cont. Easy: u satisfies ave prop!

$$\frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\,d\theta = \lim_{n\to\infty} \cdots u_n \cdots = \lim_{n\to\infty} u_n(a) = u(a)$$

To see derivatives $\rightrightarrows$ too, differentiate under Poisson formula for discs. Restrict to smaller disc. Cover by finitely many smaller discs, etc.

Remark: $P(z,e^{i\theta}) = \frac{1}{2\pi}\frac{1-|z|^2}{|z-e^{i\theta}|^2} = \operatorname{Re}\left[\frac{1}{2\pi}\,\frac{z+e^{i\theta}}{z-e^{i\theta}}\right]$

$$\int_0^{2\pi} P(z,e^{i\theta})\,u(e^{i\theta})\,d\theta = \operatorname{Re}\frac{1}{2\pi}\int_0^{2\pi}\frac{z+e^{i\theta}}{z-e^{i\theta}}\,u(e^{i\theta})\,\frac{ie^{i\theta}}{ie^{i\theta}}\,d\theta$$

$$= \operatorname{Re}\frac{1}{2\pi i}\int_{C_1(0)}\frac{1}{z-w}\,\underbrace{\frac{1}{w}\,u(w)}_{\text{cont on } C_1(0)}\,\Big[\overset{\text{can factor out}}{z}+w\Big]\,dw$$

HWK 2: p. 1. The $\int$ part is just the Cauchy integral of a cont fcn + z times such a thing.

Remark: See an operator: $u \mapsto v =$ harm conj

"Hilbert transform"

Weak averaging property A continuous fcn u on a domain Ω satisfies the WAveProp if, for each pt $a \in \Omega$, there exists $r > 0$ (which might depend on a) such that $D_r(a) \subset \Omega$ and

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + \rho e^{i\theta})\, d\theta$$

for $0 < \rho < r$.

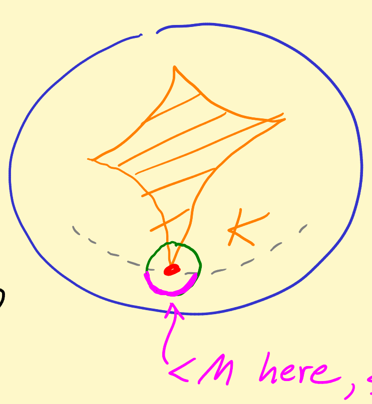
Thm WAveProp $\Longrightarrow$ harmonic

Pf This is a "local" thm. Can assume u is continuous on $D_R(0) = \Omega$, $R > 1$. Let $\tilde{u}(z) = \int_0^{2\pi} P(z, e^{i\theta}) u(e^{i\theta})\, d\theta$.

Know $\tilde{u}$ is cont on $\overline{D_1(0)}$, harmonic inside, and $\tilde{u} \equiv u$ on $C_1(0)$.

Let $U = \tilde{u} - u$. Let

$M = \underset{\overline{D_1(0)}}{\mathrm{Max}}\ \tilde{u} - u$. Suppose $M > 0$



$|z|$ is cont on K. Max is assumed at some $w_0 \in K$.

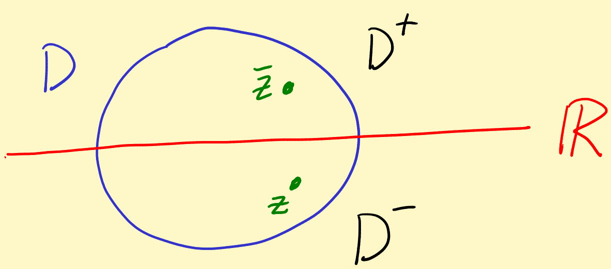
Let $K = \{z \in D_1(0) : \bar{U}(z) = M\}$.

$K \subset D_1(0)$ since $\bar{U} = 0$ on $C_1(0)$. K is closed by continuity of $\bar{U}$. So K is <u>compact</u>. $\lightning$ shows $M \leq 0$.

Repeat using $u - \tilde{u}$ in place of $\tilde{u} - u$. See $u - \tilde{u} \leq 0$ too.

So $u = \tilde{u} \leftarrow$ harmonic!

<u>Schwarz reflection principle for harmonic fcns</u>



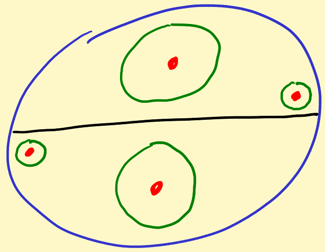
Suppose u is a real valued harmonic fcn on D^+ that is continuous up to $\mathbb{R}$ and that <u>vanishes</u> on $\mathbb{R}$.

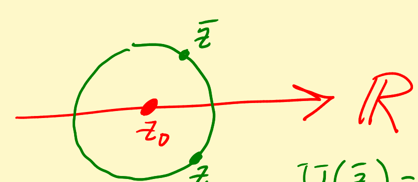
Then
$$\bar{U}(z) = \begin{cases} u(z) & \text{on } D^+ \\ 0 & \text{on } \mathbb{R} \cap D \\ -u(\bar{z}) & \text{on } D^- \end{cases}$$

is harmonic on D.

<u>Pf</u>: Note that $-u(\bar{z})$ is harmonic on D^-.

$$\left[\Delta\left[-u(x,-y) \right] = -(\Delta u)(x,-y) \equiv 0. \right]$$

$\bar{U}$ 

Last step: 

$\bar{U}(\bar{z}) = -\bar{U}(z)$

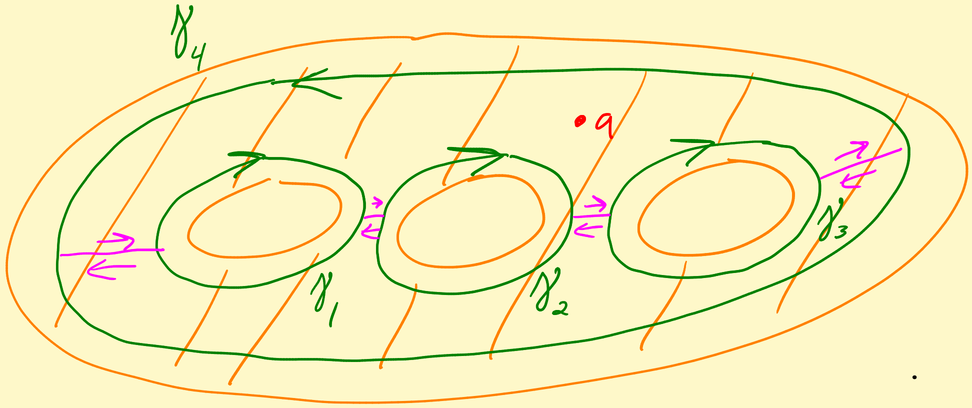
Note: Assumptions $\Rightarrow \bar{U}$ is continuous on D.

WAveProp $\Rightarrow \bar{U}$ is harmonic.

So $\int$ on top half cancels $\int$ on bottom half!

Aha! Ave $\bar{U} = 0 = \bar{U}(z_0)$.

Next time: General Cauchy theorem.



f analytic

$$\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_4$$

$$\int_\Gamma f \, dz = \sum_{k=1}^{4} \int_{\gamma_k} f \, dz = 0$$

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} \, dz$$