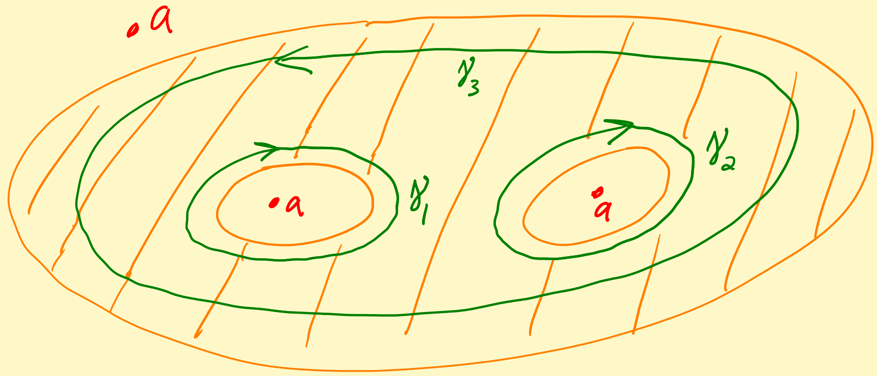
# Lecture 38 The general Cauchy theorem




$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ $$\int_\Gamma f \, dz = \sum_{k=1}^{3} \int_{\gamma_k} f \, dz = 0$$

$\Gamma$ is a concatination of closed curves. Called a "cycle".

$$\operatorname{Ind}_\Gamma(a) = \sum \operatorname{Ind}_{\gamma_k}(a) = \sum \frac{1}{2\pi i} \int_{\gamma_k} \frac{1}{z-a} \, dz$$

Fact: $\operatorname{Ind}_\Gamma(a)$ = constant on connected components of $\mathbb{C} - \operatorname{tr}(\Gamma)$. In fact, it is an integer.

Why: $$\frac{d}{da} \operatorname{Ind}_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{(z-a)^2} \, dz = \frac{1}{2\pi i} \sum \underbrace{\int_{\gamma_k} \frac{d}{dz}\left[\frac{-1}{z-a}\right] dz}_{=0}$$

because $\gamma_k$ is closed

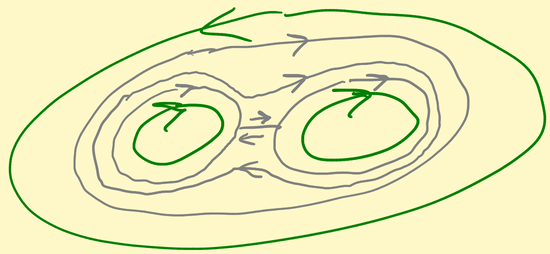
Big hypothesis to get general Cauchy thm:

$$\operatorname{Ind}_\Gamma(a) = 0 \text{ for all } a \in \mathbb{C} - \Omega.$$

Say $\Gamma$ is "homologous to zero" in $\Omega$.

Write $\Gamma \sim 0$ or $\Gamma \sim_\Omega 0$.

Think: Deform $\Gamma$ down to a pt:

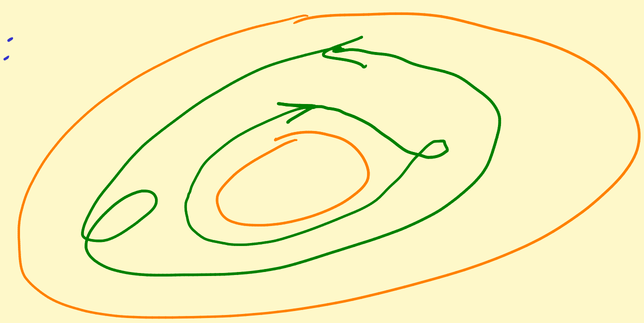


General Cauchy thm $f$ analytic on $\Omega$, $\Gamma$ a cycle in $\Omega$, $\Gamma \sim 0$. Then

1) $\int_\Gamma f\, dz = 0$

2) $f(a)\,\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$, $a \notin tr(\Gamma)$.

EX:



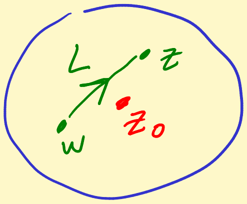
Can even go around more than once, can cross themselves and each other.

Pf: (Ahlfors. due to Beardon)

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$

Step 1 $g$ is continuous on $\Omega \times \Omega$.

Clear at $(z_0, w_0)$, $z_0 \neq w_0$. Only need to check at $(z_0, z_0)$.




$$\left| g(z,w) - g(z_0,z_0) \right| = \left| \underbrace{\frac{f(z)-f(w)}{z-w}}_{\dfrac{\int_L f'(\zeta)\,d\zeta}{z-w}} - \underbrace{f'(z_0)}_{\dfrac{\int_L f'(z_0)\,d\zeta}{z-w}} \right|$$

$$= \frac{1}{|z-w|} \underbrace{\left| \int_L \left[ f'(\zeta) - f'(z_0) \right] d\zeta \right|}_{\le \left( \underbrace{\max_L \left| f'(\zeta) - f'(z_0) \right|}_{\to 0 \text{ as radius of disc } \to 0 \text{ because } f' \text{ cont.}} \right) |z-w|}$$

Step 2 Define

$$h(z) = \int_\Gamma g(z,w)\,dw$$

Plan: Show $h$ extends to $\mathbb{C}$ as an entire fcn. It tends to zero at $\infty$. Liouville's $\Longrightarrow$ $h \equiv 0$ $\leftarrow$ that's the Cauchy thm.

Step 3 $h$ is analytic on $\Omega$.

a) Clear that $h$ is continuous on $\Omega$.

[Why: $g$ cont on $\Omega \times \Omega$. $tr(\Gamma) = \bigcup tr(\gamma_k)$ is compact. $g$ cont on compact $\overline{D_\rho(z_0)} \times tr(\Gamma)$, so is <u>uniformly</u> cont there.

$$\left| h(z_1) - h(z_2) \right| = \left| \int_\Gamma g(z_1,w) - g(z_2,w)\, dw \right|$$

$$\leq \left( \operatorname*{Max}_{w\in\Gamma} \left| g(z_1,w) - g(z_2,w) \right| \right) \text{Length}(\Gamma)$$

Use <u>uniform</u> continuity.

b) $\Delta \subset \Omega$ triangle. $\int_\Delta h\, dz =$

$$\int_\Delta \left( \int_\Gamma g(z,w)\, dw \right) dz$$

$$\underset{\uparrow \text{ Fubini}}{=} \int_\Gamma \underbrace{\left( \int_\Delta g(z,w)\, dz \right)} dw$$

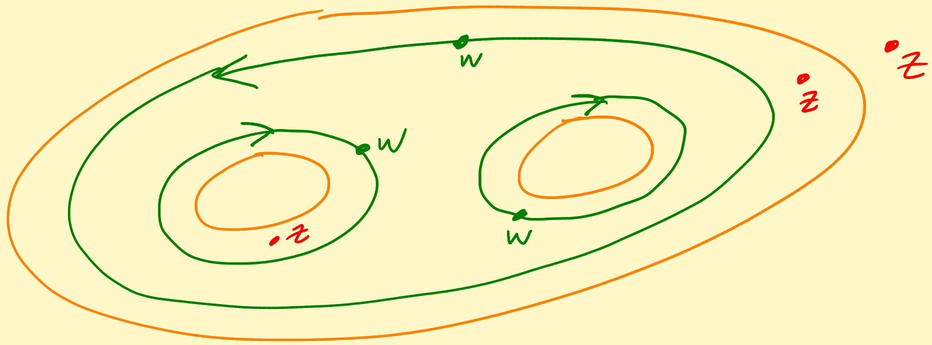
Aha! When $w$ is <u>fixed</u>,

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$

has a removable sing at $z=w$!

So $\int_\Delta = 0$.

Morera's $\Rightarrow$ $h$ analytic.




$$h(z) = \int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw$$

$$= f(z)\underbrace{\int_\Gamma \frac{1}{z-w}\,dw}_{=0} - \int_\Gamma \frac{f(w)}{z-w}\,dw$$

z outside $\Omega$
by hyp

Aha! Define $\tilde{h}(z) = -\int_\Gamma \frac{f(w)}{z-w}\,dw$ on

$$\tilde{\Omega} = \{ z \in \mathbb{C} - tr(\Gamma) \text{ where } \mathrm{Ind}_\Gamma(z) = 0 \}.$$

<u>Facts</u>: 1) $\tilde{\Omega}$ is an open set.

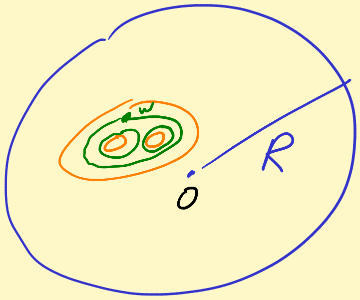
2) $\Omega \cup \tilde{\Omega} = \mathbb{C}$ because $\Gamma \sim 0$.

3) $\tilde{h}$ is analytic on $\tilde{\Omega}$ (Hwk 2: #1).

4) $h = \tilde{h}$ on $\Omega \cap \tilde{\Omega}$

So $\tilde{h}$ extends $h$ to $\mathbb{C}$ as an entire fcn!

Claim: $h \to 0$ at $\infty$.




$\mathrm{Ind}_\Gamma(a) = 0$ on $\mathbb{C} - \overline{D_R(0)}$

$z$

$$\underset{\tilde{h}}{\overset{\parallel}{|h(z)|}} = \left| - \int_\Gamma \frac{f(w)}{z-w}\, dw \right| \leq \frac{\left( \max_\Gamma |f| \right)}{|z| - R} \cdot \mathrm{Length}(\Gamma)$$

$\longrightarrow 0$ as $z \to \infty$.

Done $\frac{1}{2\pi i} \int_\Gamma \frac{f(z) - f(w)}{z - w}\, dw \equiv 0 \qquad z \notin tr(\Gamma)$

$$f(z) \underbrace{\frac{1}{2\pi i} \int_\Gamma \frac{1}{z-w}\, dw}_{\mathrm{Ind}_\Gamma(z)} = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z-w}\, dw$$

Next time: Cauchy integral formula $\Longrightarrow$ Cauchy thm.