Lecture 40 Mittag-Leffler and Weierstraß theorems

Meromorphic functions : poles at a discrete set of pts in Ω.

Give mero fcn f on Ω the value ∞ at poles. Get $f: \Omega \to \hat{\mathbb{C}}$ "analytic" from the point of view of $\hat{\mathbb{C}}$ as a complex manifold (Riemann surface).

[Coordinate chart at ∞ : $\{z : |z| > \frac{1}{\varepsilon}\} \cup \{\infty\} \xrightarrow{\frac{1}{z}} D_\varepsilon(0)$]

Mittag-Leffler thm Given a seq $\{a_n\}_{n=1}^\infty$ of distinct pts in $\mathbb{C}$ such that $a_n \to \infty$ as $n \to \infty$ and principal parts

$$R_n(z) = \frac{A_{N_n,n}}{(z-a_n)^{N_n}} + \cdots + \frac{A_{1,n}}{(z-a_n)},$$

there exists a meromorphic fcn on $\mathbb{C}$ with poles precisely at the a_n with princ part $R_n(z)$ at a_n.

EX: $a_n = n$, $R_n(z) = \dfrac{1}{(z-n)^2}$

$$\sum_{n=1}^\infty \frac{1}{(z-n)^2} \quad \text{works.}$$

On $D_R(0)$ =

$$\underbrace{\sum_{n \le R} \frac{1}{(z-n)^2}}_{\text{has given p.p. inside } D_R(0)} + \underbrace{\sum_{n > R} \frac{1}{(z-n)^2}}_{\text{converges unif on compact subsets of } D_R(0) \text{ to analytic.}}$$

[Compare with $\sum_{n>R} \frac{1}{(n-R)^2}$]

EX: Ouch! Can't do it so simply if $a_n = n$, $R_n(z) = \frac{1}{z-n}$ because $\sum \frac{1}{n} = \infty$.

Pf of M-L thm: First try $\sum_{n=1}^{\infty} R_n(z)$. Might not work. Second try: $f = \sum_{n=1}^{\infty} [R_n(z) - p_n(z)]$

where $p_n(z)$ is Taylor poly about $z=0$ for $R_n(z)$ such that $|R_n(z) - p_n(z)| < \frac{1}{2^n}$ on $\overline{D_{r_n}(0)}$ where

$r_n = \frac{|a_n|}{2}$.

Claim: This f works. Take big disc $D_R(0)$.

$\exists\ N$ such that $r_n = \frac{|a_n|}{2} > R$ if $n > N$.

$$f(z) = \underbrace{\sum_{n=1}^{N} \left(R_n(z) - p_n(z)\right)}_{\substack{\text{rational fcn with} \\ \text{desired p.p.'s in } D_R(0)}} + \sum_{n=N+1}^{\infty} \underbrace{\left(R_n(z) - p_n(z)\right)}_{\substack{|\ \ | < \frac{1}{2^n} \\ \text{on } D_R(0).}}$$

Weierstraß M-test $\Rightarrow \sum$ conv to an analytic fcn on $D_R(0)$.

Remark M-L thm true for mero fcns on any domain.

Weierstraß thm Given a seq $\{a_n\}_{n=1}^{\infty}$ of distinct pts in $\mathbb{C}$ with $a_n \to \infty$ as $n \to \infty$ and positive integers m_n, there is an entire fcn with zeroes presisely at a_n's with given multiplicities.

<u>Pf</u>: M-L thm + Gen^l residue thm $\Longrightarrow$ W. thm.

$$\left[\text{Hmmm.} \quad \frac{f'}{f} = \frac{m_n}{z-a_n} + (\text{analytic near } a_n)\right.$$

M-L Thm $\Longrightarrow$ $\exists$ mero fcn with poles at a_n's and p.p's $\frac{m_n}{z-a_n}$

$$\left[\text{Hmmm. } f(z) \text{ "="} \ e^{\log f} = e^{\int_{\gamma_a^z} \frac{f'}{f}\,dw}\right.$$

Details: M-L $\Longrightarrow$ $\exists$ mero F on $\mathbb{C}$ with poles at a_n with p.p. $\frac{m_n}{z-a_n}$. Fix a pt $a \in \mathbb{C} - \{a_n\}_{n=1}^{\infty}$.

"Define" $$f(z) = \exp\left(\int_{\gamma_a^z} F(w)\,dw\right)$$ $\leftarrow$ non-vanishing on $\mathbb{C}-\{a_n\}_1^\infty$

where γ_a^z is curve in $\mathbb{C} - \{a_n\}_{n=1}^{\infty}$ that starts at a and goes to z.

<u>Step 1</u> $f(z)$ is <u>well</u> defined.

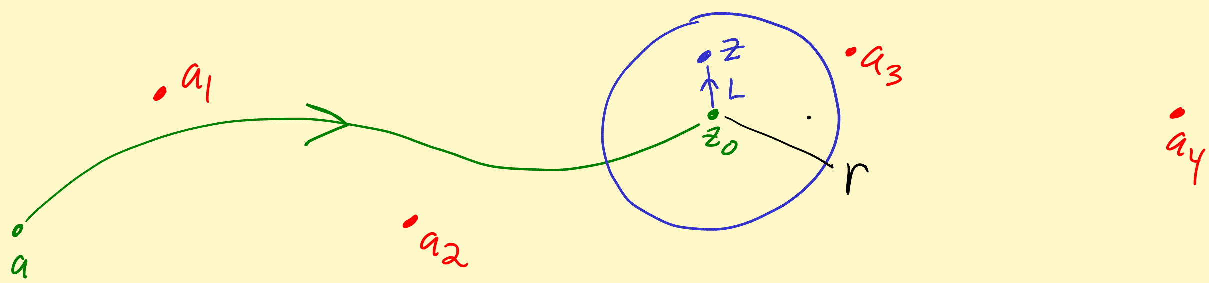
$$\frac{f(z)}{\tilde{f}(z)} = \exp\left(\underbrace{\left(\int_{\gamma_a^z} + \int_{-\tilde{\gamma}_a^z}\right)}_{\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z) \text{ is closed!}} F(w)\,dw\right)$$

$$= \exp\left(2\pi i \sum_{1}^{\infty} \underbrace{\mathrm{Ind}_{\Gamma}(a_n)}_{\text{integer!}} \cdot \underbrace{\mathrm{Res}_{a_n} F}_{m_n} \right)$$

$$= \exp\left((2\pi i)(\text{integer}) \right) = \underline{1}$$

So $f(z) = \tilde{f}(z)$ is indep of choice of γ_a^z.

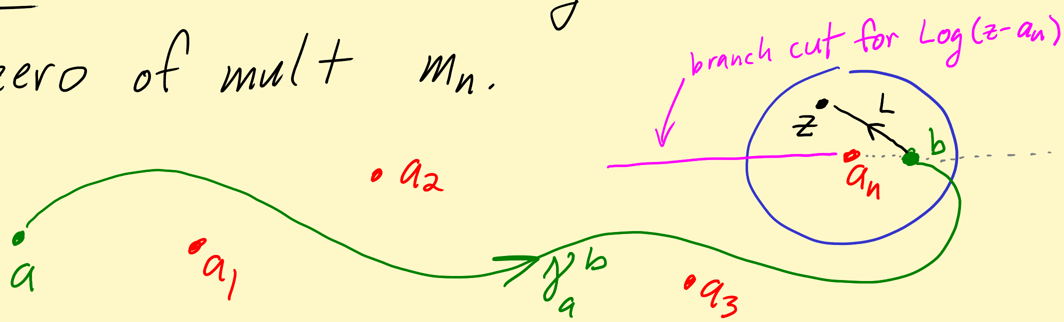
<u>Step 2</u> f is analytic on $\mathbb{C} - \{a_n\}_{n=1}^{\infty}$.



$$f(z) = \exp\left(\underbrace{\int_{\gamma_a^{z_0}} F(w)\,dw}_{\text{const}} + \underbrace{\int_{L_{z_0}^{z}} F(w)\,dw}_{\substack{\text{analytic}\\ \text{antiderivative}\\ \text{for } F \text{ on } D_r(z_0)}} \right)$$

✓ f analytic on $D_r(z_0)$.

<u>Step 3</u> a_n removable sing for f. In fact a_n is a zero of mult m_n.



$$f(z) = \exp\left(\underbrace{\int_{\gamma_a^b} F(w)\,dw}_{\text{const}} + \underbrace{\int_{L_b^z} \frac{m_n}{w-a_n}\,dw}_{m_n\left(\mathrm{Log}(z-a_n) - \underbrace{\mathrm{Log}(b-a_n)}_{\text{const}}\right)} + \underbrace{\int_{L_b^z} \underbrace{H(w)}_{\text{analytic}}\,dw}_{\text{analytic antider for } H} \right)$$

Aha! $$f(z) = \underbrace{\exp\left(m_n \mathrm{Log}(z-a_n)\right)}_{(z-a_n)^{m_n}} \underbrace{\exp\left(\begin{matrix}\text{analytic fcn}\\ \text{near } a_n\end{matrix}\right)}_{\text{non-vanishing}}$$

Hmmm. What about purple branch cut?

Aha! Argument gives $f(z)$ above cut, below cut, extend continuously up to cut and agree there.

Cor of Morera $\Rightarrow$ the extension is analytic across cut. Formulas now show a_n is removable and is a zero of mult m_n.

Weierstraß pf was via <u>infinite</u> <u>products</u>.

<u>Cor</u>: Mero fcns on $\mathbb{C} = \left\{ \frac{f}{g} : f, g \text{ entire}, g \not\equiv 0 \right\}$