MATH 530 Exam solutions

1. Define $\log z = \operatorname{Ln}|z| + i\theta$ where $\theta \in \arg z$ with $-\frac{3\pi}{2} < \theta < \frac{\pi}{2}$. Then $\int_\gamma \frac{1}{z}\,dz = \int_\gamma \left(\frac{d}{dz}\log z\right) dz$

$= \log(-1+i) - \log(2+2i) = \left[\operatorname{Ln}\sqrt{2} - \frac{5\pi}{4}i\right] - \left[\operatorname{Ln} 2\sqrt{2} + i\frac{\pi}{4}\right]$

$= -\operatorname{Ln} 2 - \frac{3\pi}{2}i$.

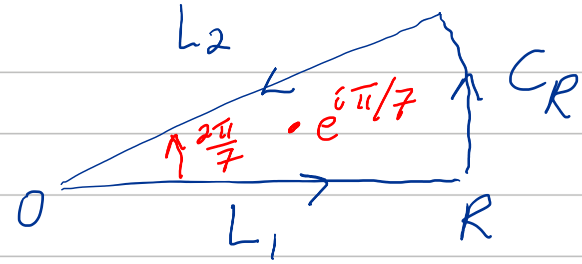
2. Suppose z_0 is a limit point of the zero set of ∇u in Ω. Since Ω is open, there is a disc $D_r(z_0)$ contained in Ω. Let v be a harmonic conjugate for u on $D_r(z_0)$ and define $f = u + iv$.

The proof of the C-R Eqns gave that $f' = u_x + i v_x = u_x - i u_y$ and we see that z_0 is a limit point of the zero set of f'.

The Identity Theorem implies that $f' \equiv 0$ on $D_r(z_0)$. So f, and therefore u, is constant on $D_r(z_0)$.

u is therefore constant on Ω (by the Theorem that states that if a harmonic function on a domain vanishes on a disc, it vanishes on the domain).

3.



$$f(z) = \frac{z}{z^7+1}$$

$$\left| \int_{C_R} f(z)\,dz \right| \le \left(\operatorname*{Max}_{C_R} |f| \right) \left(\frac{2\pi}{7} R \right)$$

$$\le \frac{R}{R^7 - 1} \cdot \frac{2\pi}{7} R \quad \text{if } R > 1$$

$$\longrightarrow 0 \text{ as } R \to \infty.$$

$$\int_{L_1} f\,dz = \int_0^R \frac{t}{t^7+1}\,dt \quad \leftarrow \text{want. Call it } I_R.$$

$$-L_2 : z(t) = t e^{i2\pi/7}, \quad 0 \le t \le R$$

$$\int_{L_2} f(z)\,dz = -\int_{-L_2} f(z)\,dz = -\int_0^R \frac{t e^{i2\pi/7}}{(t e^{i2\pi/7})^7 + 1} \underbrace{e^{i2\pi/7}\,dt}_{z'(t)\,dt}$$

$$= -e^{i4\pi/7} I_R$$

Residue Thm: $\left(\int_{L_1} + \int_{L_2} + \int_{C_R} \right) f\,dz = 2\pi i \operatorname{Res}_{e^{i\pi/7}} f(z)$

$$\left(1 - e^{i4\pi/7}\right) I_R + \int_{C_R} f\,dz = 2\pi i \frac{e^{i\pi/7}}{7 (e^{i\pi/7})^6}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\left(1 - e^{i4\pi/7}\right) I + 0 = \frac{2\pi i\, e^{i\pi/7}}{7 e^{i6\pi/7}}$$

So $$I = \frac{2\pi i\, e^{-i5\pi/7}}{7\left(1 - e^{i4\pi/7}\right)} \cdot \frac{e^{-i2\pi/7}}{e^{-i2\pi/7}} = \frac{\pi}{7} \frac{2i(-1)}{\left(e^{-i2\pi/7} - e^{i2\pi/7}\right)} = \frac{\pi}{7 \sin\frac{2\pi}{7}}$$

4. a) Since $|f(z)| \to \infty$ as $|z| \to \infty$, there is an $R > 0$ such that $|f(z)| > 1$ if $|z| > R$, and f has no zeroes outside $C_R(0)$. $\overline{D_R(0)}$ is compact, and so if f had infinitely many zeroes, there would exist a limit point of zeroes in $\overline{D_R(0)}$, and this would imply that $f \equiv 0$ ↯ by the Identity Theorem. So f has at most finitely many zeroes. b) Let $F = \frac{1}{f} - ($Sum of the principal parts of $1/f$ at zeroes of $f)$. Note that F has removable singularities at zeroes of f, so we may think of F as an entire function. Also note that terms like $\frac{A_m}{(z-a)^m}$ in the principal parts tend to zero as $z \to \infty$ (and $\frac{1}{f}$ does too). Hence, $|F(z)| \to 0$ as $|z| \to \infty$ and F is a <u>bounded</u> entire function. Liouville's

$\Rightarrow$ $F \equiv c$, a constant. Therefore, f is rational, i.e., $f(z) = \dfrac{P(z)}{Q(z)}$ where P and Q are complex polynomials. We may remove common factors from P and Q [by the Fund. Thm. Algebra]. Since f is analytic, and since $P(z) \neq 0$ where $Q(z) = 0$, we conclude that Q has no zeroes, i.e., that Q is constant. Hence $f = (\text{Const})\,P$ is a polynomial.