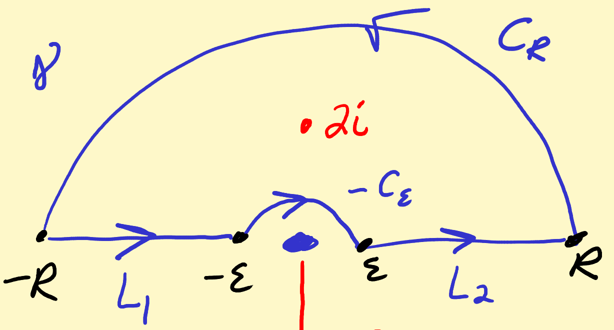
Solutions to Exam 2

1.



$$f(z) = \frac{\log z}{(z^2+4)^2}$$

$\log z = \operatorname{Ln}|z| + i\theta$, $\theta \in \arg z$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$$\int_{L_2} f(z)\,dz = \int_\varepsilon^R \frac{\operatorname{Ln} t}{(t^2+4)^2}\,dt = I_\varepsilon^R \longleftarrow \text{want}$$

$-L_1$: $z(t) = -t$, $\varepsilon \le t \le R$. $z'(t) = -1$

$$\int_{L_1} = -\int_{-L_1} = -\int_\varepsilon^R \frac{\operatorname{Ln}|-t| + i\pi}{((-t)^2+4)^2}(-1)\,dt = \underbrace{\int_\varepsilon^R \frac{\operatorname{Ln} t}{(t^2+4)^2}\,dt}_{I_\varepsilon^R} + i\pi \int_\varepsilon^R \frac{dt}{(t^2+4)^2}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \left(\max_{0 \le t \le \pi} \frac{|\operatorname{Ln} R + it|}{(R^2-4)^2}\right)(\pi R) \le \frac{\operatorname{Ln} R + \pi}{(R^2-4)^2}\cdot \pi R$$

$|\operatorname{Ln} R + it| \le \operatorname{Ln} R + t$

$|z^2+4|^2 \ge \big||z^2| - 4\big|^2$

$\longrightarrow 0$ as $R \to \infty$

$$\left|\int_{-C_\varepsilon} f(z)\,dz\right| \le \frac{|\operatorname{Ln}\varepsilon| + \pi}{(4-\varepsilon^2)^2}\cdot(\pi\varepsilon) \longrightarrow 0 \text{ as } \varepsilon \to 0$$

$$\Big(\underset{\downarrow}{\int_{L_1}} + \int_{L_2} + \underset{\downarrow 0}{\int_{C_{-\varepsilon}}} + \underset{\downarrow 0}{\int_{C_R}}\Big) f(z)\,dz = 2\pi i \operatorname{Res}_{2i} f(z)$$

$$2I + i\pi \underbrace{\int_0^\infty \frac{dt}{(t^2+4)^2}}_{\text{don't care about this because we can take real parts to get } 2I} = 2\pi i \, \mathrm{Res}_{2i} \left[\frac{1}{(z-2i)^2} \cdot \underbrace{\frac{\log z}{(z+2i)^2}}_{G(z) = A_0 + A_1(z-2i) + \cdots} \right]$$

$$= 2\pi i \, A_1 = 2\pi i \, \frac{G'(2i)}{1!}$$

$$= 2\pi i \left. \frac{\frac{1}{z} \cdot (z+2i)^2 - (\log z)\, 2(z+2i)}{\left((z+2i)^2\right)^2} \right|_{z=2i}$$

$$= 2\pi i \, \frac{\frac{1}{2i}(4i)^2 - \left(\ln 2 + i\frac{\pi}{2}\right)(8i)}{256}$$

$$= \frac{\pi}{128}\left(-8 + 8\ln 2 + 4\pi i\right)$$

$$2I = \text{Real part of } 2\pi i \, \mathrm{Res}_{2i} f(z) = \frac{\pi}{16}(\ln 2 - 1)$$

$$I = \frac{\pi}{32}(\ln 2 - 1)$$

2. $f(z) = a_N z^N + a_{N+1} z^{N+1} + \cdots = z^N H(z)$

where $H(0) = a_N = \dfrac{f^{(N)}(0)}{N!}$.

Let
$$F(z) = \begin{cases} \dfrac{f(z)}{z^N} = H(z) & z \neq 0 \\[2ex] \dfrac{f^{(N)}(0)}{N!} & z = 0 \end{cases}$$

F has a removable sing at $z = 0$.

Pick $z_0 \in D_1(0)$ and r with $|z_0| < r < 1$.

Max princ $\Rightarrow$ $|F(z_0)| \leq \max\limits_{|z|=r} \frac{|f(z)|}{|z|^N} < \frac{1}{r^N}$.

Let $r \nearrow 1$ to see $|F(z_0)| \leq 1$. z_0 arbitrary $\Rightarrow$ $|F(z)| \leq 1$ on $D_1(0)$. So $|F(0)| = \left|\frac{f^{(N)}(0)}{N!}\right| \leq 1$ and get $|f^{(N)}(0)| \leq N!$. If equality occurs, $|F(0)| = 1$ assumes a local max at 0. Max princ $\Rightarrow$ $F(z) \equiv \lambda$, a const. $|F(0)| = |\lambda|$. So $|\lambda| = 1$. Finally, $\frac{f(z)}{z^N} \equiv \lambda$ if $z \neq 0$. So

$f(z) = \lambda z^N$ if $z \neq 0$. True at $z=0$ too.

3. Let $F(z) = z f(z)$ for $z \neq 0$. Note that

$$|F(z)| \leq |z| \frac{C}{|z|^\alpha} = C|z|^{1-\alpha} \to 0 \text{ as } z \to 0$$

since $0 < \alpha < 1$.

Hence F has a removable sing at $z=0$ and has a zero of some order $N \geq 1$ at $z=0$.

Hence $F(z) = z^N H(z)$ for some H analytic on

$D_1(0)$ with $H(0) \neq 0$.

Now $zf(z) = z^N H(z)$ yields that $z \neq 0$

$$\underline{f(z) = z^{N-1} H(z)} \quad \text{for } z \neq 0.$$

and we see that f has a removable sing at $z = 0$.

4. Let $g(z) = f(z) - z$ and $h(z) = z$. $|z| = 1$ on $C_1(0)$

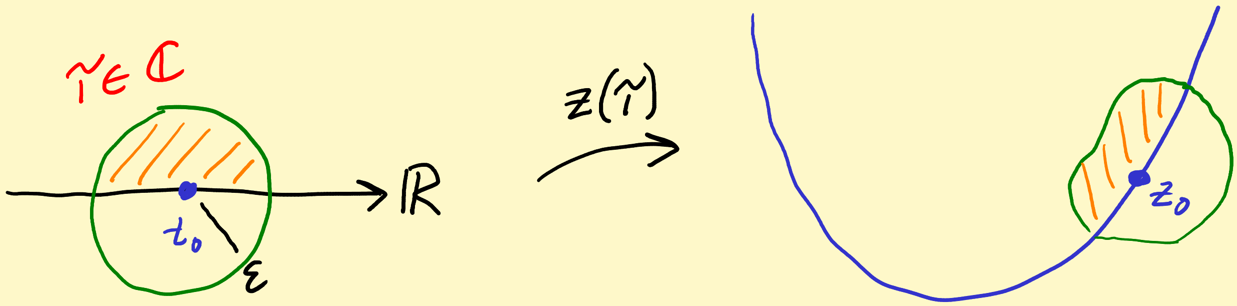
Notice that $|h(z) - g(z)| = |f(z)| < \underbrace{|z|}_{|h(z)|} = 1$

on $C_1(0)$. Rouché's $\Rightarrow$ g and h have same number of zeroes inside $C_1(0)$, namely, exactly one. So f has exactly one fixed point.

Note: Since $|z| = 1$ on $C_1(0)$, can make this problem more challenging by stating it: f analytic on $D_R(0)$, $R > 1$ and $|f(z)| < 1$ when $|z| = 1$. Show f has exactly one fixed point in $D_1(0)$.

5. $z(t) = t + it^2$ parametrizes the parabola

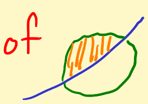
$z'(t) = 1 + 2it \neq 0$, so $z(\tau)$ is locally 1-1.



Shrink ε so that $z(\tilde{\tau})$ is one-to-one analytic onto a domain about z_0 and sends the upper half disc above the parabola. (The bug test shows this is true locally. Can shrink disc to make this true on the disc.) Let $\tilde{\tau}(z)$ denote the inverse mapping, which is analytic on domain containing z_0.

Now $f(z(\tilde{\tau}))$ satisfies hyp of Schwarz reflection on top half disc, so extends analytically via analytic fcn $F(\tilde{\tau})$ on disc. $F(\tilde{\tau}) = f(z(\tilde{\tau}))$. Let $\tilde{\tau} = \tilde{\tau}(z)$ to see that

$$f(z) = \underbrace{F(\tilde{\tau}(z))}_{\text{extends to bottom half of}}$$



So f extends analytically past z_0.