Math 530
Exam

Each problem is worth 20 points.

1. The first part of the Schwarz lemma says that, if f is an analytic function mapping the unit disc into itself such that $f(0) = 0$, then $|f(z)| \leq |z|$ for $z \in D_1(0)$ and $|f'(0)| \leq 1$. What does the second part of the lemma say?

2. If u is a real valued $\mathcal{C}^2$-smooth harmonic function on a domain Ω, prove that
$$f(x + iy) := \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$
defines an analytic function f on Ω.

3. Show that the Cauchy-Goursat theorem for triangles implies the Cauchy theorem for pentagons.

4. State Liouville's theorem and show how to prove it using the Cauchy estimates and Taylor's formula for power series coefficients.

5. Explain how to use the Residue theorem to compute
$$I = \int_0^\infty \frac{t^2}{t^4 + 1}\, dt$$

by integrating the complex function $f(z) = z^2/(z^4 + 1)$ around the quarter circle of radius R in the first quadrant with base along the real axis from 0 to R, letting $R \to \infty$, and taking limits. If you claim a limit exists, prove that it does. Find the value of the integral I and confirm that it is a real number.