Solutions to Exam 1

1. ... and if $|f(z)| = |z|$ for some $z \neq 0$, **or** if $|f'(0)| = 1$, then $f(z) = \lambda z$ where λ is a constant of modulus 1.

2. u C^2-smooth $\Rightarrow$ $\underbrace{\frac{\partial u}{\partial x}}_{\bar{U}}$ and $\underbrace{-\frac{\partial u}{\partial y}}_{\bar{V}}$ are C^1-smooth.

Claim: $\bar{U}$ and $\bar{V}$ satisfy the C-R equs.

$$\frac{\partial \bar{U}}{\partial x} = u_{xx} \overset{?}{=} -u_{yy} = \frac{\partial \bar{V}}{\partial y}$$

yes, because $\Delta u \equiv 0$.

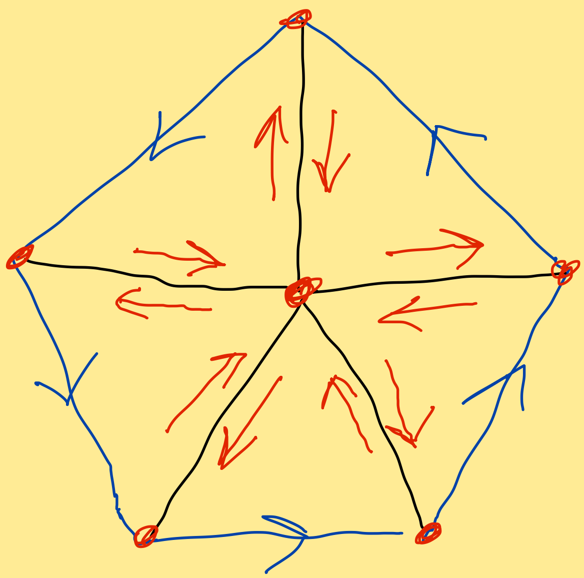
$$\frac{\partial \bar{U}}{\partial y} = u_{yx} \overset{?}{=} u_{xy} = -\frac{\partial \bar{V}}{\partial x}$$

yes, because u C^2-smooth

$\bar{U}, \bar{V}$ C^1-smooth and satisfy C-R eqns

$\Rightarrow \bar{U} + i\bar{V}$ analytic. ✓

3.



Sum of counterclockwise integrals around the five triangles: the integrals along the black lines cancel, leaving counterclockwise integral around pentagon. So $\int_{\Delta\text{'s}} h\,dz = 0$ implies $\int_{\text{pentagon}} h\,dz = 0$.

4. Liouvile's : A bounded entire fcn must be constant. Suppose f is analytic on $\mathbb{C}$ and $|f(z)| \leq M$ for all z.

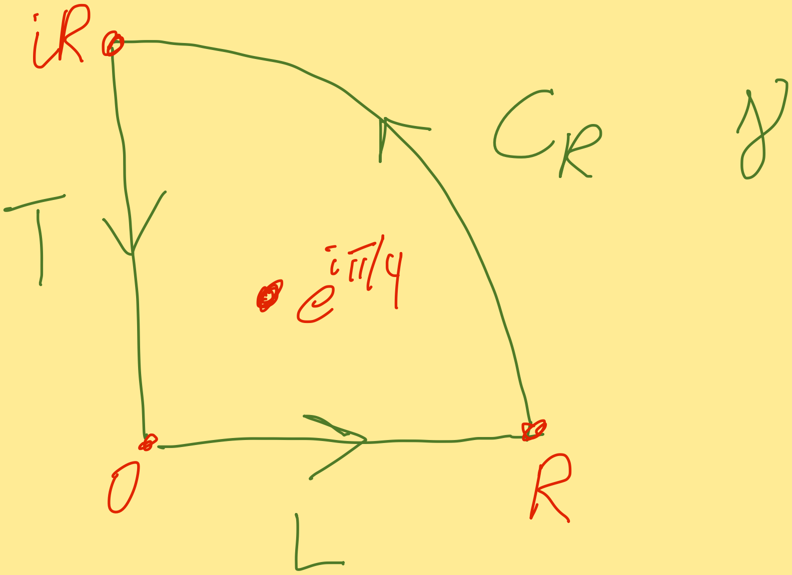
f is given by a convergent power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Taytor's formula plus Cauchy estimates yields

$$|a_n| = \left| \frac{f^{(n)}(0)}{n!} \right| \leq \frac{1}{n!} \left[\frac{n! \max_{|z|=r} |f(z)|}{r^n} \right] \leq \frac{M}{r^n}$$

$\rightarrow 0$ as $r \rightarrow \infty$ if $n \geq 1$.

So $a_n = 0$ for $n \geq 1$. $f(z) \equiv a_0$. ✓

5.



$$f(z)=\frac{z^2}{z^4+1}$$

$L: z(t)=t, \ 0\le t\le R, \ \int_L f(z)dz=\int_0^R \frac{t^2}{t^4+1}dt$

$-T: z(t)=it, \ 0\le t\le R, \ z'(t)=i$

$$\int_T f(z)\,dz = -\int_{-T} f\,dz$$

$$= -\int_0^R \frac{(it)^2}{(it)^4+1}\cdot i\,dt$$

$$= i\int_0^R \frac{t^2}{t^4+1}\,dt$$

$$\left|\int_{C_R}\right| \le \left(\max_{C_R} \frac{|z|^2}{|z^4+1|}\right)\left(\tfrac{\pi}{2}R\right) \le \frac{R^2}{R^4-1}\left(\tfrac{\pi}{2}R\right) \text{ if } R>1.$$

$$\longrightarrow 0 \text{ as } R\to\infty$$

$$\int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_{e^{i\pi/4}} f(z)$$

$$\underbrace{\left(\int_L + \int_T\right.}\left. + \int_{C_R}\right) f(z)\,dz = 2\pi i \,\frac{\left(e^{i\pi/4}\right)^2}{4\left(e^{i\pi/4}\right)^3}$$

$$(1+i)\int_0^R \frac{t^2}{t^4+1}\,dt + \int_{C_R} f\,dz = 2\pi i \cdot \frac{e^{i\pi/2}}{4e^{i3\pi/4}} \leftarrow = i$$

$$\downarrow \qquad\qquad \downarrow$$

$$(1+i)\,I + 0 = -\frac{2\pi}{4e^{i3\pi/4}} = -\frac{\pi}{2}e^{-i3\pi/4}$$

$$I = \frac{1}{1+i}\left(-\frac{\pi}{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right)$$

$$= \frac{1}{1+i}\cdot\frac{\pi}{2}\cdot\frac{\sqrt{2}}{2}(1+i) = \frac{\pi\sqrt{2}}{4} = \underline{\underline{\frac{\pi}{2\sqrt{2}}}}$$